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NON-ISOTHERMAL THEORY OF RODS, PLATES AND SHELLS

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Abstract—A detailed development of nonlinear thermodynamical theories of rods and shells is discussed using the three dimensional theory of classical continuum mechanics as a starting point. A portion of the paper supplements and amplifies the previous work on the subject by Green, Laws and Naghdi [1]. Also, the method of approximation used in [1] is altered since it has been found to be only partly satisfactory. Special attention is given to non-isothermal linear theories of elastic shells and straight elastic rods which are derived from the three dimensional equations.

1. INTRODUCTION

THIS paper is concerned with further developments of nonlinear (as well as linear) theories of rods and shells which, in particular, supplements and amplifies the previous work on the subject by Green, Laws and Naghdi [1], referred to subsequently as I. Our starting point is again the three dimensional theory of classical continuum mechanics. The three dimensional theory is reduced to a two dimensional theory for a thermoelastic shell, or plate, and a one dimensional theory for a thermoelastic rod by using suitable representations for the displacement and the temperature. Before describing the scope of the paper, however, it is desirable to recall briefly certain closely related, but independent, recent developments in the theories of deformable surfaces and deformable curves to which frequent reference will be made throughout the paper.

Consider a two dimensional surface to every point of which one director (i.e. a vector which remains invariant in length under superposed rigid body motions) is assigned. A general thermodynamical theory of such a surface—called a Cosserat surface—has been developed by Green, Naghdi and Wainwright [2], within the framework of two dimensional generalized continua. A related contribution, limited to isothermal deformation of elastic directed surfaces, has been given by Cohen and DeSilva [3]. Similarly, consider a one dimensional curve to every point of which two directors are assigned. A general thermodynamical theory of such a deformable curve (or a Cosserat rod) has been developed by Green and Laws [4], within the framework of one dimensional generalized continua. A related contribution restricted to isothermal deformation of elastic rods has been given by Cohen [5].

Further aspects of the Cosserat surface have been discussed by Green and Naghdi (e.g. [6–9]) who, in particular, have emphasized [7, 9] the relevance and applicability of the linear theory of an elastic Cosserat surface to problems of plates and shells, regarded as three dimensional bodies. Similarly, the relevance of a linearized version of the theory of Green and Laws [4] to straight elastic rods, regarded as three dimensional bodies, has been discussed by Green, Laws and Naghdi [10].

In the present paper, we continue the previous developments in I in an effort to illuminate further our understanding of the existing theories of rods and shells, especially with regard to thermal effects and heat conduction equations. The basic three dimensional equations of classical continuum mechanics are collected in Section 2 and the remainder of the paper is so arranged that the part which deals with shells (Sections 3–9) may be read independently of the part on rods (Sections 10–14). For both shells and rods, we employ in our developments an exact representation for the expansion of the displacement and a similar exact representation for the expansion of temperature. Previously in I, an approximation for the temperature was used.

The two dimensional equations, including the energy equation, for shells in Sections 3–4 are obtained systematically from the corresponding three dimensional equations of Section 2 and appropriate entropy inequalities are deduced in Section 5 from the entropy inequality usually employed in classical continuum mechanics. Some aspects of the thermodynamic developments of Sections 3–5 were anticipated earlier in a paper by Naghdi [11]. Results for thermoelastic shells, including the constitutive equations, are stated in section 6 and provide an infinite set of equations for an infinite number of unknowns. Next, by approximation, we reduce in Section 7 the infinite set of equations to a system which is formally equivalent to that in [2] for an elastic Cosserat surface, except for more generality here in the temperature. The method of approximation is different from that used in I since it has been found that the procedure given in I is only partly satisfactory. In the next two sections (Sections 8–9) we discuss linear theories of thermoelastic shells and plates and include a detailed development of the residual energy equations for the determination of temperature. The latter, upon the neglect of thermo-mechanical coupling, are of the same form as the heat conduction equations derived by Bolotin [12].

For rods in Sections 10–11, by a procedure similar to that used for shells, we obtain the basic equations which include a one dimensional form for the energy equations. Entropy inequalities for rods can be discussed along lines similar to those in Section 5 for shells; however, this is not included in the present paper. Results for thermoelastic rods, including the constitutive equations, are stated in Section 12 and again consist of an infinite set of equations for an infinite number of unknowns. In Section 13, by approximation, we reduce the infinite set of equations to a finite set formally equivalent to that in [4] for an elastic rod, except for more generality here in the temperature. The method of approximation is different from that used in I since it has been found that the procedure given in I is only partly satisfactory. Finally, in Section 14, we discuss a non-isothermal linear theory of straight elastic rods in detail and include the derivation of appropriate heat conduction equations.

2. NOTATION AND FORMULAE

Points of a three dimensional continuum are defined by a general convected coordinate system θ^i. Covariant and contravariant base vectors at points of the continuum at time t are denoted by $\mathbf{g}_i$, $\mathbf{g}^i$ with corresponding metric tensors g_{ij}, g^{ij}, Latin indices having values 1, 2, 3.

Thus

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \qquad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \qquad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j,$$
$$\mathbf{g}_i = \frac{\partial \mathbf{r}^*}{\partial \theta^i}, \qquad \mathbf{r}^* = \mathbf{r}^*(\theta^1, \theta^2, \theta^3, t), \tag{2.1}$$

where $\mathbf{r}^*$ is the position vector of a typical particle θ^i and δ^i_j is the Kronecker delta. The velocity vector $\mathbf{v}^*$ at time t is given by

$$\mathbf{v}^* = \frac{D\mathbf{r}^*}{Dt} = \dot{\mathbf{r}}^*, \tag{2.2}$$

where D/Dt or a superposed dot denotes the material time derivative (holding θ^i fixed).

The energy equation for an arbitrary material volume V bounded by a surface A at time t is

$$\frac{D}{Dt}\int_V (U^* + \tfrac{1}{2}\mathbf{v}^* \cdot \mathbf{v}^*)\rho^* \, \mathrm{d}V = \int_V (r^* + \mathbf{f}^* \cdot \mathbf{v}^*)\rho^* \, \mathrm{d}V + \int_A (\mathbf{t} \cdot \mathbf{v}^* - h^*) \, \mathrm{d}A \tag{2.3}$$

where ρ^* is density, U^* is internal energy per unit mass, $\mathbf{f}^*$ is the body force per unit mass, r^* is the heat supply function per unit mass per unit time, $\mathbf{t}$ is the stress vector and h^* the heat flux across A.

With the help of invariance conditions under superposed rigid body motions we can deduce from (2.3) equations of mass conservation, momentum and moment of momentum. Alternatively these equations may be regarded as separate postulates. For our present purpose these equations may be written in the point form

$$\rho^* g^{1/2} = k(\theta^1, \theta^2, \theta^3) \qquad g = \det g_{ij}, \tag{2.4}$$

$$\mathbf{T}_{i,i} + \rho^* \mathbf{f}^* g^{\frac{1}{2}} = \rho^* \dot{\mathbf{v}}^* g^{\frac{1}{2}}, \tag{2.5}$$

$$\mathbf{g}_i \times \mathbf{T}_i = 0, \tag{2.6}$$

where a comma denotes partial differentiation with respect to θ^i, k is independent of t, and the stress vector $\mathbf{t}$ across a surface whose unit normal is $\mathbf{u}$ is given by

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{g^{\frac{1}{2}}}, \qquad \mathbf{u} = u_i \mathbf{g}^i = u^i \mathbf{g}_i. \tag{2.7}$$

Also

$$\mathbf{T}_i = g^{\frac{1}{2}} \tau^{ij} \mathbf{g}_j, \tag{2.8}$$

where τ^{ij} is the symmetric contravariant stress tensor.

With the help of (2.4)–(2.8) the energy equation can be reduced to the point form

$$\rho^* r^* - \rho^* \dot{U}^* + \tau^{ij} \dot{\gamma}_{ij} - q^{*k}{}_{|k} = 0, \tag{2.9}$$

where

$$h^* = u_k q^{*k}, \qquad \dot{\gamma}_{ij} = \tfrac{1}{2}\dot{g}_{ij}, \tag{2.10}$$

and a vertical line denotes covariant differentiation using Christoffel symbols defined from the metric tensor g_{ij}. For later use, (2.9) may be written in the alternative form

$$\rho^* r^* - \rho^* \dot{U}^* + \frac{\mathbf{T}_i \cdot \dot{\mathbf{v}}^*_{,i}}{g^{\frac{1}{2}}} - \frac{(q^{*k} g^{\frac{1}{2}})_{,k}}{g^{\frac{1}{2}}} = 0. \tag{2.11}$$

To complete our system of equations we need the entropy production inequality

$$\frac{D}{Dt}\int_V \rho^* S^* \, \mathrm{d}V - \int_V \frac{\rho^* r^*}{T^*} \, \mathrm{d}V + \int_A \frac{h^*}{T^*} \, \mathrm{d}A \geq 0, \tag{2.12}$$

where S^* is entropy per unit mass and $T^*(>0)$ is the absolute temperature. In point form (2.12) is

$$\rho^* T^* \dot{S}^* - \rho^* r^* + \frac{T^*}{g^{\frac{1}{2}}}\left(\frac{q^{*k} g^{\frac{1}{2}}}{T^*}\right)_{,k} \geq 0. \tag{2.13}$$

Finally we observe that when given surface forces $\mathbf{P}$ are applied at the boundary surface of the body, measured per unit area of the body at time t, then

$$\mathbf{t} = \mathbf{P} \tag{2.14}$$

at the boundary.

We require generalized forms of the energy equation and entropy production inequality and these can be obtained from (2.11) and (2.13). Let ϕ be a scalar function of the coordinates θ^i. For example, one form of ϕ to be used later in shell theory is $(\theta^3)^n$. We multiply (2.11) by ϕ and integrate throughout an arbitrary volume V. After some straightforward manipulation we obtain

$$\begin{aligned}\frac{D}{Dt}\int_V \left(U^* + \frac{1}{2}\mathbf{v}^* \cdot \mathbf{v}^*\right)\rho^* \phi \, \mathrm{d}V = \int_V (r^* + \mathbf{f}^* \cdot \mathbf{v}^*)\rho^* \phi \, \mathrm{d}V + \int_A (\mathbf{t} \cdot \mathbf{v}^* - h^*)\phi \, \mathrm{d}A \\ - \int_V [\mathbf{T}_i \cdot \mathbf{v}^* - q^{*i} g^{\frac{1}{2}}]\frac{\partial \phi}{\partial \theta^i}\frac{\mathrm{d}V}{g^{\frac{1}{2}}} = 0.\end{aligned} \tag{2.15}$$

Again, let ψ be a scalar function of the coordinates θ^i which is such that

$$\psi \geq 0 \tag{2.16}$$

throughout some closed region V. We multiply (2.13) by ψ and integrate throughout V to get

$$\frac{D}{Dt}\int_V \rho^* S^* \psi \, \mathrm{d}V - \int_V \frac{\rho^* r^* \psi}{T^*} \, \mathrm{d}V + \int \frac{h^* \psi}{T^*} \, \mathrm{d}A - \int_V \frac{q^{*i}}{T^*}\frac{\partial \psi}{\partial \theta^i} \, \mathrm{d}V \geq 0. \tag{2.17}$$

3. SHELLS

We adopt the convention that Greek indices take the values 1, 2 and write

$$\theta^3 = \xi. \tag{3.1}$$

The parametric equation $\xi = 0$ defines a surface $\mathscr{s}$ in space at time t, which we assume to be smooth and non-intersecting, the position vector of any point of $\mathscr{s}$ being given by

$$\mathbf{r} = \mathbf{r}(\theta^1, \theta^2, t) = \mathbf{r}^*(\theta^1, \theta^2, 0, t). \tag{3.2}$$

We assume that the continuum is bounded by the surfaces

$$\xi = \alpha, \qquad \xi = \beta \; (\alpha < 0 < \beta) \tag{3.3}$$

which are non-intersecting with themselves, each other, or with $\mathscr{s}$, and which are such that $\mathscr{s}$ lies entirely between them. We fix the relation of the surface $\mathscr{s}$ to the bounding surface (3.3) by imposing the condition

$$\int_\alpha^\beta \rho^* g^{\frac{1}{2}} \xi \, d\xi = \int_\alpha^\beta k(\theta^1, \theta^2, \xi)\xi \, d\xi = 0. \tag{3.4}$$

The above condition is independent of time, so that once $\mathscr{s}$ is determined by such an equation (in, say, a reference state) it remains so determined. We also assume that the continuum is bounded by a surface

$$f(\theta^1, \theta^2) = 0 \tag{3.5}$$

which is such that ξ = constant are closed smooth curves on this surface. We call such a continuum a shell.

We assume that the position vector and temperature of any point of the shell at time t can be represented by the expressions

$$\mathbf{r}^* = \mathbf{r}(\theta^1, \theta^2, t) + \sum_{N=1}^{\infty} \xi^N \mathbf{d}_N, \tag{3.6}$$

$$T^* = T_0(\theta^1, \theta^2, t) + \sum_{N=1}^{\infty} \xi^N T_N, \tag{3.7}$$

where $\mathbf{d}_N$ are vector functions and T_N are scalar functions of θ^1, θ^2, t. We also assume that the series (3.6) and (3.7) may be differentiated as many times as required with respect to any of its variables at least in the open region $\alpha < \xi < \beta$.

From (2.2) and (3.6) we have

$$\mathbf{v}^* = \mathbf{v} + \sum_{N=1}^{\infty} \xi^N \mathbf{w}_N, \tag{3.8}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{w}_N = \dot{\mathbf{d}}_N. \tag{3.9}$$

For later convenience we put

$$\mathbf{d} = \mathbf{d}_1, \qquad \mathbf{w} = \mathbf{w}_1. \tag{3.10}$$

We call the vectors $\mathbf{d}_N$ directors and $\mathbf{w}_N$ director velocities and observe that $\mathbf{d}_N$ are unchanged in length when the shell is subjected to superposed rigid body motions. From (2.1) and

(3.6) we see that

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \sum_{N=1}^{\alpha} \xi^N \frac{\partial \mathbf{d}_N}{\partial \theta^\alpha},$$
$$\mathbf{g}_3 = \sum_{N=1}^{\alpha} N\xi^{N-1}\mathbf{d}_N, \qquad \mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}. \tag{3.11}$$

We find it convenient to use the notation

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \qquad a = \det a_{\alpha\beta},$$
$$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \qquad a^{\alpha\lambda}a_{\lambda\beta} = \delta^\alpha_\beta, \tag{3.12}$$

and we denote the unit normal vector to the surface (3.2) by $\mathbf{a}_3$, a vector function of θ^1, θ^2, t. In three dimensional theory of continuum mechanics it is usual to assume that

$$[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] > 0 \tag{3.13}$$

for all time and all values of θ^i. In particular, it is valid for $\xi = 0$ so that, from (3.11) and (3.10),

$$[\mathbf{a}_1\mathbf{a}_2\mathbf{d}] > 0. \tag{3.14}$$

With the help of (3.8), the energy equation (2.3) was reduced to two dimensional form in a previous paper† I. We quote the final result and refer readers to I for further details. Thus

$$\frac{D}{Dt}\int \rho\left(U + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \sum_{N=2}^{\infty} k^N \mathbf{w}_N \cdot \mathbf{v} + \frac{1}{2}\sum_{M,N=1}^{\infty} k^{M+N}\mathbf{w}_M \cdot \mathbf{w}_N\right) d\sigma$$
$$= \int \rho\left(r + \mathbf{F}\cdot\mathbf{v} + \sum_{N=1}^{\infty} \mathbf{L}^N \cdot \mathbf{w}_N\right) d\sigma + \oint \left(\mathbf{N}\cdot\mathbf{v} + \sum_{N=1}^{\infty} \mathbf{M}^N \cdot \mathbf{w}_N - h\right) dc. \tag{3.15}$$

The surface integral in (3.15) is over an arbitrary element of the surface $\mathscr{s}$ bounded by some closed curve c and the line integrals are around c. Also

$$\rho a^{\frac{1}{2}} = \int_\alpha^\beta \rho^* g^{\frac{1}{2}}\, d\xi = \int_\alpha^\beta k\, d\xi,$$
$$\rho k^N a^{\frac{1}{2}} = \int_\alpha^\beta \xi^N k\, d\xi \qquad (N \geq 2), \tag{3.16}$$

$$\rho r a^{\frac{1}{2}} = \int_\alpha^\beta r^* k\, d\xi - [h^*(gg^{33})^{\frac{1}{2}}]_{\xi=\alpha} - [h^*(gg^{33})^{\frac{1}{2}}]_{\xi=\beta}, \tag{3.17}$$

$$\rho \mathbf{F} a^{\frac{1}{2}} = \int_\alpha^\beta \mathbf{f}^* k\, d\xi + [\mathbf{t}(gg^{33})^{\frac{1}{2}}]_{\xi=\alpha} + [\mathbf{t}(gg^{33})^{\frac{1}{2}}]_{\xi=\beta} = \int_\alpha^\beta \mathbf{f}^* k\, d\xi + [\mathbf{T}_3]_\alpha^\beta, \tag{3.18}$$

$$\rho \mathbf{L}^N a^{\frac{1}{2}} = \int_\alpha^\beta \mathbf{f}^* k\xi^N\, d\xi + [\mathbf{t}\xi^N(gg^{33})^{\frac{1}{2}}]_{\xi=\alpha} + [\mathbf{t}\xi^N(gg^{33})^{\frac{1}{2}}]_{\xi=\beta}$$
$$= \int_\alpha^\beta \mathbf{f}^* k\xi^N\, d\xi + [\xi^N \mathbf{T}_3]_\alpha^\beta, \tag{3.19}$$

$$\rho U a^{\frac{1}{2}} = \int_\alpha^\beta U^* k\, d\xi, \tag{3.20}$$

† As already noted in section 1, throughout the present paper, I refers to Ref. [1].

where

$$[\psi(\theta^\alpha, \xi, t)]_\alpha^\beta = \psi(\theta^\alpha, \beta, t) - \psi(\theta^\alpha, \alpha, t). \tag{3.21}$$

In addition, if $\mathbf{v}$ is the outward unit normal to a curve c on the surface $\mathscr{s}$ and

$$\mathbf{v} = v_\alpha \mathbf{a}^\alpha, \tag{3.22}$$

then

$$\mathbf{N} = \mathbf{N}^\alpha v_\alpha, \qquad \mathbf{M}^N = \mathbf{M}^{N\alpha} v_\alpha, \qquad h = q^\alpha v_\alpha, \tag{3.23}$$

where

$$\mathbf{N}^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta \mathbf{T}_\alpha \, \mathrm{d}\xi, \tag{3.24}$$

$$\mathbf{M}^{N\alpha} a^{\frac{1}{2}} = \int_\alpha^\beta \xi^N \mathbf{T}_\alpha \, \mathrm{d}\xi, \qquad \mathbf{M}^{0\alpha} = \mathbf{N}^\alpha, \tag{3.25}$$

$$q^\alpha a^{\frac{1}{2}} = \int_\alpha^\beta q^{*\alpha} g^{\frac{1}{2}} \, \mathrm{d}\xi. \tag{3.26}$$

We now substitute

$$\phi = \xi^n \qquad (n = 1, 2, \ldots) \tag{3.27}$$

in (2.15) and follow a procedure similar to that used in obtaining (3.15) from (2.3). Thus

$$\begin{aligned} \frac{D}{Dt} \int \rho \left[U^n + \frac{1}{2} k^n \mathbf{v} \cdot \mathbf{v} + \sum_{N=1}^{\infty} k^{N+n} \mathbf{w}_N \cdot \mathbf{v} + \frac{1}{2} \sum_{M,N=1}^{\infty} k^{M+N+n} \mathbf{w}_M \cdot \mathbf{w}_N \right] \mathrm{d}\sigma \\ = \int \rho \left[r^n + R^n + \mathbf{L}^n \cdot \mathbf{v} + \sum_{N=1}^{\infty} \mathbf{L}^{N+n} \cdot \mathbf{w}_N - \mathbf{m}^n \cdot \mathbf{v} - \sum_{N=1}^{\infty} \frac{n}{N+n} \mathbf{m}^{N+n} \cdot \mathbf{w}_N \right] \mathrm{d}\sigma \\ + \oint \left[\mathbf{M}^n \cdot \mathbf{v} + \sum_{N=1}^{\infty} \mathbf{M}^{N+n} \cdot \mathbf{w}_N - h^n \right] \mathrm{d}c, \end{aligned} \tag{3.28}$$

where, in addition to quantities already defined, we have

$$\rho r^n a^{\frac{1}{2}} = \int_\alpha^\beta r^* k \xi^n \, \mathrm{d}\xi - [\xi^n h^* (g g^{33})^{\frac{1}{2}}]_{\xi=\alpha} - [\xi^n h^* (g g^{33})^{\frac{1}{2}}]_{\xi=\beta}, \tag{3.29}$$

$$\rho U^n a^{\frac{1}{2}} = \int_\alpha^\beta U^* k \xi^n \, \mathrm{d}\xi, \tag{3.30}$$

$$\mathbf{m}^N a^{\frac{1}{2}} = N \int_\alpha^\beta \xi^{N-1} \mathbf{T}_3 \, \mathrm{d}\xi \qquad (N \geq 0), \tag{3.31}$$

$$\rho R^n a^{\frac{1}{2}} = n \int_\alpha^\beta q^{*3} \xi^{n-1} g^{\frac{1}{2}} \, \mathrm{d}\xi, \tag{3.32}$$

$$q^{n\alpha} a^{\frac{1}{2}} = \int_\alpha^\beta q^{*\alpha} \xi^n g^{\frac{1}{2}} \, \mathrm{d}\xi, \qquad h^n = q^{n\alpha} v_\alpha. \tag{3.33}$$

4. FURTHER DEVELOPMENT OF ENERGY EQUATIONS

Equation (3.15) may be regarded as a special case of (3.28) when $n = 0$. In point form, equation (3.28) becomes

$$\begin{aligned}&\rho r^n+\rho R^n-\rho \dot{U}^n+(\mathbf{M}^{n\alpha}{}_{|\alpha}+\rho\bar{\mathbf{L}}^n-\mathbf{m}^n)\cdot\mathbf{v}+\mathbf{M}^{n\alpha}\cdot\mathbf{v}_{,\alpha}\\&+\sum_{N=1}^{\infty}\left[\mathbf{M}^{N+n\alpha}{}_{|\alpha}+\rho\bar{\mathbf{L}}^{N+n}-\frac{n}{N+n}\mathbf{m}^{N+n}\right]\cdot\mathbf{w}_N+\sum_{N=1}^{\infty}\mathbf{M}^{N+n\alpha}\cdot\mathbf{w}_{N,\alpha}-q^{n\alpha}{}_{|\alpha}=0,\end{aligned}\tag{4.1}$$

where a comma denotes partial differentiation with respect to θ^α and a vertical line now denotes surface covariant differentiation using Christoffel symbols derived from the surface metric tensor $a_{\alpha\beta}$. Also,

$$\begin{aligned}\bar{\mathbf{F}}&=\bar{\mathbf{L}}^0=\mathbf{F}-\dot{\mathbf{v}}-\sum_{N=2}^{\infty}k^N\dot{\mathbf{w}}_N,\\ \bar{\mathbf{L}}^N&=\mathbf{L}^N-k^N\dot{\mathbf{v}}-\sum_{M=1}^{\infty}k^{M+N}\dot{\mathbf{w}}_M\qquad(N\geq 2),\\ \bar{\mathbf{L}}&=\bar{\mathbf{L}}^1=\mathbf{L}-\sum_{M=1}^{\infty}k^{M+1}\dot{\mathbf{w}}_M.\end{aligned}\tag{4.2}$$

We consider a motion of the shell at time t in which the velocities differ from the given velocities only by superposed uniform translational rigid body velocities and we assume that these do not change the quantities r^n, R^n, U^n, $\mathbf{M}^{n\alpha}$, $\mathbf{m}^n$, $\bar{\mathbf{L}}^n$, $q^{n\alpha}$ for $n \geq 0$. From (4.1), we then deduce the results

$$\mathbf{N}^{\alpha}{}_{|\alpha}+\rho\bar{\mathbf{F}}=0,\qquad \mathbf{M}^{n\alpha}{}_{|\alpha}+\rho\bar{\mathbf{L}}^n-\mathbf{m}^n=0\qquad(n=1,2,\ldots).\tag{4.3}$$

With the help of (4.3), equation (4.1) reduces to

$$\rho r^n+\rho R^n-\rho\dot{U}^n+\mathbf{M}^{n\alpha}\cdot\mathbf{v}_{,\alpha}+\sum_{N=1}^{\infty}\frac{N}{N+n}\mathbf{m}^{N+n}\cdot\mathbf{w}_N+\sum_{N=1}^{\infty}\mathbf{M}^{N+n\alpha}\cdot\mathbf{w}_{N,\alpha}-q^{n\alpha}{}_{|\alpha}=0\qquad(n=0,1,2,\ldots).\tag{4.4}$$

Next we consider a motion of the shell in which the velocities differ from the given velocities only by a superposed uniform rigid body angular velocity, the shell having the same orientation as before. Then, assuming that r^n, R^n, U^n, $\mathbf{M}^{n\alpha}$, $\mathbf{m}^n$, $\bar{\mathbf{L}}^n$, $q^{n\alpha}$ are unaltered by such rigid body motions we deduce, from (4.4), the equation

$$\mathbf{M}^{n\alpha}\times\mathbf{a}_\alpha+\sum_{N=1}^{\infty}\left(\frac{N}{N+n}\mathbf{m}^{N+n}\times\mathbf{d}_N+\mathbf{M}^{N+n\alpha}\times\mathbf{d}_{N,\alpha}\right)=0\quad\text{for } n=0,1,\ldots.\tag{4.5}$$

The case $n = 0$ can more conveniently be written as

$$\mathbf{N}^\alpha\times\mathbf{a}_\alpha+\sum_{N=1}^{\infty}(\mathbf{m}^N\times\mathbf{d}_N+\mathbf{M}^{N\alpha}\times\mathbf{d}_{N,\alpha})=0\tag{4.6}$$

which was obtained previously in I but in component form.

It is of interest to observe that the equations of motion (4.3) can also be obtained directly from (2.5) and (3.8); and the equations (4.5) and (4.6) can be deduced from (2.6) and (3.11).

With the help of (4.5) the system of energy equations (4.4) can be simplified. For this purpose, we introduce the following quantities:

$$\begin{aligned} \mathbf{d}_N &= d_{Ni}\mathbf{a}^i = d_N{}^i\mathbf{a}_i, \qquad \partial\mathbf{d}_N/\partial\theta^\alpha = \lambda_{Ni\alpha}\mathbf{a}^i, \\ \mathbf{d}_1 &= \mathbf{d} = d_i\mathbf{a}^i = d^i\mathbf{a}_i, \qquad \partial\mathbf{d}/\partial\theta^\alpha = \lambda_{i\alpha}\mathbf{a}^i, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\lambda_{N\beta\alpha} = d_{N\beta|\alpha} - b_{\beta\alpha}d_{N3}, \qquad \lambda_{N3\alpha} = d_{N3,\alpha} + b^\beta_\alpha d_{N\beta}, \\ &\lambda_{N\cdot\alpha}^{\,\beta} = a^{\beta\gamma}\lambda_{N\gamma\alpha}, \qquad \lambda_{N\cdot\alpha}^{\,3} = \lambda_{N3\alpha}, \\ &\lambda_{1\beta\alpha} = \lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\beta\alpha}d_3, \qquad \lambda_{13\alpha} = \lambda_{3\alpha} = d_{3,\alpha} + b^\beta_\alpha d_\beta, \\ &\lambda^\beta_{\cdot\alpha} = a^{\beta\gamma}\lambda_{\gamma\alpha}, \qquad \lambda^3_{\cdot\alpha} = \lambda_{3\alpha}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\mathbf{N}^\alpha = N^{i\alpha}\mathbf{a}_i, \qquad \mathbf{m}^N = m^{Ni}\mathbf{a}_i, \qquad \mathbf{m}^1 = \mathbf{m} = m^i\mathbf{a}_i, \\ &\mathbf{M}^{N\alpha} = M^{Ni\alpha}\mathbf{a}_i, \qquad \mathbf{M}^{1\alpha} = \mathbf{M}^\alpha = M^{i\alpha}\mathbf{a}_i. \end{aligned} \tag{4.9}$$

Then, using (4.5), equations (4.4) reduce to

$$\rho r^n + \rho R^n - \rho\dot{U}^n + M'^{n\beta\alpha}\eta_{\alpha\beta} + \sum_{N=1}^{\infty}\frac{N}{N+n}m^{N+ni}\dot{d}_{Ni} + \sum_{N=1}^{\infty} M^{N+ni\alpha}\dot{\lambda}_{Ni\alpha} - q^{n\alpha}{}_{|\alpha} = 0, \tag{4.10}$$

where

$$\eta_{\alpha\beta} = \dot{e}_{\alpha\beta}, \qquad 2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{4.11}$$

$A_{\alpha\beta}$ being the initial value of $a_{\alpha\beta}$ and

$$M'^{n\beta\alpha} = M'^{n\alpha\beta} = M^{n\beta\alpha} - \sum_{N=1}^{\infty}\left(\frac{N}{N+n}m^{N+n\alpha}d_{N\cdot}^{\,\beta} + M^{N+n\alpha\gamma}\lambda_{N\cdot\gamma}^{\,\beta}\right). \tag{4.12}$$

The first part of equation (4.12) arises from the component form of (4.5), the remaining components giving

$$M^{n3\alpha} + \sum_{N=1}^{\infty}\frac{N}{N+n}(m^{N+n3}d_{N\cdot}^{\,\alpha} - m^{N+n\alpha}d_{N\cdot}^{\,3}) + \sum_{N=1}^{\infty}(M^{N+n3\gamma}\lambda_{N\cdot\gamma}^{\,\alpha} - M^{N+n\alpha\gamma}\lambda_{N\cdot\gamma}^{\,3}) = 0. \tag{4.13}$$

Equations (4.12), (4.13) hold for $n = 0, 1, 2, \ldots$ but it is more convenient to write the equations corresponding to $n = 0$ in the forms

$$N'^{\beta\alpha} = N'^{\alpha\beta} = N^{\beta\alpha} - \sum_{N=1}^{\infty}(M^{N\alpha\gamma}\lambda_{N\cdot\gamma}^{\,\beta} + m^{N\alpha}d_{N\cdot}^{\,\beta}), \tag{4.14}$$

$$N^{3\alpha} + \sum_{N=1}^{\infty}(m^{N3}d_{N\cdot}^{\,\alpha} - m^{N\alpha}d_{N\cdot}^{\,3}) + \sum_{N=1}^{\infty}(M^{N3\gamma}\lambda_{N\cdot\gamma}^{\,\alpha} - M^{N\alpha\gamma}\lambda_{N\cdot\gamma}^{\,3}) = 0. \tag{4.15}$$

At this point we introduce the three dimensional Helmholtz free energy function A^* by the equation

$$A^* = U^* - T^*S^*. \tag{4.16}$$

Then writing

$$\rho S^n a^{\frac{1}{2}} = \int_\alpha^\beta S^* k \xi^n \, d\xi, \tag{4.17}$$

$$\rho A^n a^{\frac{1}{2}} = \int_\alpha^\beta A^* k \xi^n \, d\xi, \tag{4.18}$$

we have, from (3.7) and (4.16),

$$A^n = U^n - \sum_{N=0}^{\infty} S^{N+n} T_N. \tag{4.19}$$

Equation (4.10) now becomes

$$\begin{aligned} &\rho r^n + \rho R^n - \rho\left[\dot{A}^n + \sum_{N=0}^{\infty} (\dot{S}^{N+n} T_N + S^{N+n} \dot{T}_N)\right] - q^{n\alpha}{}_{|\alpha} + M'^{n\beta\alpha} \eta_{\alpha\beta} \\ &+ \sum_{N=1}^{\infty} \left(\frac{N}{N+n} m^{N+ni} \dot{d}_{Ni} + M^{N+ni\alpha} \dot{\lambda}_{Ni\alpha}\right) = 0. \end{aligned} \tag{4.20}$$

5. ENTROPY INEQUALITIES

We first recall that $\psi \geq 0$ in the entropy inequality (2.17) and set

$$\psi = (-\alpha + \xi)^n \qquad (\alpha \leq \xi \leq \beta). \tag{5.1}$$

In addition, remembering that $T^* > 0$, we put

$$\Theta^* = \frac{1}{T^*} = \sum_{N=0}^{\infty} \xi^N \Theta_N \tag{5.2}$$

so that, in view of (3.7),

$$T_0 \Theta_0 = 1, \qquad \sum_{N=0}^{r} T_N \Theta_{r-N} = 0 \qquad (r = 1, 2, \ldots). \tag{5.3}$$

If we substitute (5.1) and (5.2) into (2.17) and reduce the resulting equation to point form we obtain the inequality

$$\begin{aligned} &\rho \sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \left[\dot{S}^r - \sum_{N=0}^{\infty} r^{r+N} \Theta_N\right] + \sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \sum_{N=0}^{\infty} (q^{r+N\alpha}{}_{|\alpha} \Theta_N + q^{r+N\alpha} \Theta_{N,\alpha}) \\ &- \rho \sum_{r=0}^{n-1} \binom{n-1}{r} (-\alpha)^{n-r-1} \sum_{N=0}^{\infty} \frac{n}{r+N+1} R^{r+N+1} \Theta_N \geq 0. \end{aligned} \tag{5.4}$$

In view of (5.3),

$$\begin{aligned} &\dot{S}^r - \sum_{N=0}^{\infty} \Theta_N \left[\dot{A}^{r+N} + \sum_{M=0}^{\infty} \overline{S^{M+N+r} T_M}^{\cdot}\right] \\ &= \dot{S}^r - \sum_{N=0}^{\infty} \Theta_N \left[\dot{A}^{r+N} + \sum_{M=0}^{\infty} S^{M+N+r} \dot{T}_M\right] - \sum_{n=0}^{\infty} \sum_{N=0}^{n} \dot{S}^{n+r} \Theta_{n-N} T_N \\ &= -\sum_{N=0}^{\infty} \Theta_N \left[\dot{A}^{r+N} + \sum_{M=0}^{\infty} S^{M+N+r} \dot{T}_M\right]. \end{aligned} \tag{5.5}$$

Substituting for r^n from (4.20) into (5.4) and using (5.5), we finally obtain the inequalities

$$\sum_{r=0}^{n}\binom{n}{r}(-\alpha)^{n-r}\Bigg\{\sum_{N=0}^{\infty}\Theta_N\Bigg[-\rho\Bigg(\dot{A}^{r+N}+\sum_{M=0}^{\infty}S^{M+N+r}\dot{T}_M\Bigg)+M'^{r+N\beta\alpha}\eta_{\alpha\beta}$$
$$+\sum_{M=1}^{\infty}\frac{M}{M+N+r}m^{M+N+ri}\dot{d}_{Mi}+\sum_{M=1}^{\infty}M^{M+N+ri\alpha}\dot{\lambda}_{Mi\alpha}+\frac{N}{N+r}\rho R^{N+r}\Bigg]$$
$$+\sum_{N=0}^{\infty}q^{N+r\alpha}\Theta_{N,\alpha}\Bigg\}\geq 0 \quad \text{for } n=0,1,2,\ldots. \tag{5.6}$$

6. THERMOELASTIC SHELL

Two methods are available for finding constitutive relations appropriate for an elastic shell. In the first method we start with the three dimensional thermoelastic equations

$$S^* = -\frac{\partial A^*}{\partial T^*} \tag{6.1}$$

$$\tau^{rs} = \tfrac{1}{2}\rho^*\left(\frac{\partial A^*}{\partial \gamma_{rs}}+\frac{\partial A^*}{\partial \gamma_{sr}}\right) \tag{6.2}$$

where† A^* is a function of T^* and strain γ_{rs} given by

$$\gamma_{rs} = \tfrac{1}{2}(\mathbf{g}_r\cdot\mathbf{g}_s-\mathbf{G}_r\cdot\mathbf{G}_s), \tag{6.3}$$

$\mathbf{G}_r$ being the value of $\mathbf{g}_r$ in some preferred initial state. From (3.7), (3.11), (3.16) and (4.18) we see that A^n can be expressed in the form

$$A^n = A^n(T_N, e_{\alpha\beta}, d_{Mi}, \lambda_{Mi\alpha}). \tag{6.4}$$

Also A^n depends on initial values of d_{Mi}, $\lambda_{Mi\alpha}$ and T_N. In (6.4), $N, n = 0, 1, 2, \ldots$ and $M = 1, 2, \ldots$. With the help of (3.11), (3.24), (3.25), (3.31), (4.9), (4.12), (4.13), (4.16), (4.19), (6.1) and (6.2), a direct calculation yields

$$S^{n+M} = -\frac{\partial A^n}{\partial T_M} \qquad (n=0,1,2,\ldots;\ M=0,1,2,\ldots), \tag{6.5}$$

$$M'^{r\beta\alpha} = \tfrac{1}{2}\rho\left(\frac{\partial A^r}{\partial e_{\alpha\beta}}+\frac{\partial A^r}{\partial e_{\beta\alpha}}\right) \qquad (r=0,1,\ldots), \tag{6.6}$$

$$\frac{N}{N+n}m^{N+ni} = \rho\frac{\partial A^n}{\partial d_{Ni}} \qquad (n=0,1,\ldots;\ N=1,2,\ldots), \tag{6.7}$$

$$M^{N+ni\alpha} = \rho\frac{\partial A^n}{\partial \lambda_{Ni\alpha}} \qquad (n=0,1,\ldots;\ N=1,2,\ldots). \tag{6.8}$$

In addition

$$-q^{*k}T^*_{,k} \geq 0, \tag{6.9}$$

† The free energy A^* also depends on the initial metric tensor G_{rs}.

and the residual energy equation is

$$\rho^* r^* - \frac{(q^{*k} g^{\frac{1}{2}})_{,k}}{g^{\frac{1}{2}}} - \rho^* \dot{S}^* T^* = 0. \tag{6.10}$$

From (6.10) we can, by integration and use of (3.7), obtain the residual energy equations

$$\int \rho(r^n + R^n)\, d\sigma - \int \rho \sum_{M=0}^{\infty} \dot{S}^{n+M} T_M\, d\sigma - \oint h^n\, dc = 0,$$

or

$$\rho(r^n + R^n) - \rho \sum_{M=0}^{\infty} \dot{S}^{n+M} T_M - q^{n\alpha}{}_{|\alpha} = 0. \tag{6.11}$$

Using (3.7), by integration of (6.9) we obtain the inequalities

$$-\sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \sum_{N=0}^{\infty} \left[\frac{N}{r+N} \rho R^{r+N} T_N + q^{r+N\alpha} T_{N,\alpha} \right] \geq 0 \qquad (r = 0, 1, \ldots). \tag{6.12}$$

By noting that (6.9) is equivalent to

$$q^{*k} \Theta^*_{,k} \geq 0,$$

instead of (6.12), we can deduce the equivalent inequalities

$$\sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \sum_{N=0}^{\infty} \left[\frac{N}{r+N} \rho R^{r+N} \Theta_N + q^{r+N\alpha} \Theta_{N,\alpha} \right] \geq 0 \qquad (n = 0, 1, \ldots). \tag{6.13}$$

Alternatively, we may start with the inequality (5.6) and suitable constitutive assumptions and again deduce all the results (6.5)–(6.13) but we omit details.

In addition to equations of the present section, we have equations of motion (4.3), and equations (4.14) and (4.15) which give values for $N^{\alpha\beta}$ and $N^{3\alpha}$. Also we have the system of equations (4.12) and (4.13) for values of $n = 1, 2 \ldots$. Equations (4.12)–(4.15) essentially arise from the symmetry of the three dimensional stress tensor; and in view of (6.6)–(6.8) and the fact that A^n in (6.4) is evaluated from (4.18) with the help of (3.11), the set of equations (4.12) and (4.13) is satisfied identically. As a result there is some redundancy in the equations we have obtained for a thermoelastic shell and we summarize the essential equations below. The equations of motion (4.3) in component form are

$$\begin{aligned} N^{\beta\alpha}{}_{|\alpha} - b^\beta_\alpha N^{3\alpha} + \rho \bar{F}^\beta &= 0, \\ N^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} N^{\beta\alpha} + \rho \bar{F}^3 &= 0, \end{aligned} \tag{6.14}$$

$$\begin{aligned} M^{\beta\alpha}{}_{|\alpha} - b^\beta_\alpha M^{3\alpha} + \rho \bar{L}^\beta &= m^\beta, \\ M^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} M^{\beta\alpha} + \rho \bar{L}^3 &= m^3, \end{aligned} \tag{6.15}$$

$$\begin{aligned} M^{n\beta\alpha}{}_{|\alpha} - b^\beta_\alpha M^{n3\alpha} + \rho \bar{L}^{n\beta} &= m^{n\beta}, \\ M^{n3\alpha}{}_{|\alpha} + b_{\alpha\beta} M^{n\beta\alpha} + \rho \bar{L}^{n3} &= m^{n3}, \end{aligned} \qquad (n \geq 2) \tag{6.16}$$

$$\begin{aligned} &N^{3\alpha} + \sum_{N=1}^{\infty} (m^{N3} d_{N\cdot}^{\ \alpha} - m^{N\alpha} d_{N\cdot}^{\ 3}) + \sum_{N=1}^{\infty} (M^{N3\gamma} \lambda_{N\cdot\gamma}^{\ \alpha} - M^{N\alpha\gamma} \lambda_{N\cdot\gamma}^{\ 3}) = 0, \\ &N'^{\beta\alpha} = N'^{\alpha\beta} = N^{\beta\alpha} - \sum_{N=1}^{\infty} (M^{N\alpha\gamma} \lambda_{N\cdot\gamma}^{\ \beta} + m^{N\alpha} d_{N\cdot}^{\ \beta}). \end{aligned} \tag{6.17}$$

Writing A for A^0 the constitutive equations are

$$S^N = -\frac{\partial A}{\partial T_N} \qquad (N = 0, 1, \ldots), \tag{6.18}$$

and

$$N'^{\beta\alpha} = \frac{1}{2}\rho\left(\frac{\partial A}{\partial e_{\alpha\beta}} + \frac{\partial A}{\partial e_{\beta\alpha}}\right),$$

$$m^{Ni} = \rho\frac{\partial A}{\partial d_{Ni}} \qquad (N = 1, 2, \ldots), \tag{6.19}$$

$$M^{Ni\alpha} = \rho\frac{\partial A}{\partial \lambda_{Ni\alpha}} \qquad (N = 1, 2, \ldots),$$

$$A = A(T_M, e_{\alpha\beta}, d_{Ni}, \lambda_{Ni\alpha}) \qquad (M = 0, 1, \ldots; N = 1, 2, \ldots). \tag{6.20}$$

The residual energy equations are

$$\rho(r^n + R^n) - \sum_{N=0}^{\infty} \dot{S}^{n+N} T_N - q^{n\alpha}{}_{|\alpha} = 0 \qquad (n = 0, 1, \ldots), \tag{6.21}$$

and the entropy inequalities are

$$-\sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \sum_{N=0}^{\infty} \left[\frac{N}{r+N}\rho R^{r+N} T_N + q^{r+N\alpha} T_{N,\alpha}\right] \geq 0 \qquad (r = 0, 1, \ldots). \tag{6.22}$$

To complete the theory, constitutive equations are also required for R^n and $q^{n\alpha}$ but we only consider these for linear elastic isotropic plates in Section 8.

7. APPROXIMATION FOR SHELLS

In I where temperature effects were only partly taken into account, a method of approximation was suggested in order to reduce the infinite set of equations for kinematic quantities to finite form. In a recent detailed examination of the linear theory of isotropic plates we have found that the approximation procedure is only partly satisfactory and hence a different procedure is discussed here.† At this stage we make no approximation about the temperature.

We assume that the free energy function A in (6.20) can be represented by an approximate expression in terms of $e_{\alpha\beta}, d_i, \lambda_{i\alpha}, T_N (N = 0, 1, \ldots)$ only. We do not solve the problem of how to determine this approximate form of A from (6.20) which was obtained from the full three dimensional expression for the free energy. Thus, we set

$$A = A(e_{\alpha\beta}, \lambda_{i\alpha}, d_i, T_N) \qquad (N = 0, 1, \ldots), \tag{7.1}$$

approximately, where $A(e_{\alpha\beta}, \lambda_{i\alpha}, d_i, T_N)$ is a different function from that in (6.20). Using $(6.19)_{2,3}$, it follows that‡

$$m^{Ni} = 0, \qquad M^{Ni\alpha} = 0 \qquad (N \geq 2). \tag{7.2}$$

† A comparison of shell theory and classical three dimensional isothermal elasticity has been discussed recently by Sensenig [13] from another point of view.

‡ These results hold approximately, since they are obtained with the help of (7.1).

The equations of motion (6.16) are then satisfied if

$$\bar{L}^{Ni} = 0 \qquad (N \geq 2) \tag{7.3}$$

and the remaining equations of motion (6.14), (6.15) and (6.17) become

$$\begin{aligned} N^{\beta\alpha}{}_{|\alpha} - b^{\beta}_{\alpha} N^{3\alpha} + \rho \bar{F}^{\beta} &= 0, \\ N^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} N^{\beta\alpha} + \rho \bar{F}^{\beta} &= 0, \end{aligned} \tag{7.4}$$

$$\begin{aligned} M^{\beta\alpha}{}_{|\alpha} - b^{\beta}_{\alpha} M^{3\alpha} + \rho \bar{L}^{\beta} &= m^{\beta}, \\ M^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} M^{\beta\alpha} + \rho \bar{L}^{3} &= m^{3}, \end{aligned} \tag{7.5}$$

$$\begin{aligned} N^{3\alpha} + m^3 d^\alpha - m^\alpha d^3 + M^{3\gamma} \lambda^{\alpha}_{.\gamma} - M^{\alpha\gamma} \lambda^{3}_{.\gamma} &= 0, \\ N'^{\beta\alpha} = N'^{\alpha\beta} &= N^{\beta\alpha} - M^{\alpha\gamma} \lambda^{\beta}_{.\gamma} + m^\alpha d^\beta, \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} \bar{F}^{\beta} &= F^{\beta} - \dot{\mathbf{v}} \cdot \mathbf{a}^{\beta}, \qquad & \bar{F}^{3} &= F^{3} - \dot{\mathbf{v}} \cdot \mathbf{a}_{3}, \\ \bar{L}^{\beta} &= L^{\beta} - k^2 \dot{\mathbf{w}} \cdot \mathbf{a}^{\beta}, \qquad & \bar{L}^{3} &= L^{3} - k^2 \dot{\mathbf{w}} \cdot \mathbf{a}_{3}, \end{aligned} \tag{7.7}$$

approximately.

Constitutive equations are now given by

$$\begin{aligned} S^N &= -\frac{\partial A}{\partial T_N}, \\ N'^{\beta\alpha} &= \frac{1}{2}\rho\left(\frac{\partial A}{\partial e_{\alpha\beta}} + \frac{\partial A}{\partial e_{\beta\alpha}}\right), \\ M^{i\alpha} &= \rho \frac{\partial A}{\partial \lambda_{i\alpha}}, \\ m^i &= \rho \frac{\partial A}{\partial d_i}, \end{aligned} \tag{7.8}$$

and constitutive relations are still required for $q^{n\alpha}$ and R^n before the residual energy equations

$$\rho(r^n + R^n) - \rho \sum_{N=0}^{\infty} \dot{S}^{n+N} T_N - q^{n\alpha}{}_{|\alpha} = 0 \qquad (n = 0, 1, \ldots) \tag{7.9}$$

are completely specified. Such constitutive equations are subject to restrictions imposed by the inequalities (6.22).

We recall that equations (4.12) and (4.13) for $n \geq 1$ are identities in a complete theory for elastic shells. Since the expression (7.1) for the free energy A is no longer exact, we expect that some of these equations may not be satisfied by the approximation used in obtaining (7.1)–(7.9). In fact, in view of (7.2), only equations corresponding to $n = 1$ in (4.12) and (4.13) are violated and these equations reduce to

$$M'^{\beta\alpha} = M^{\beta\alpha} = M^{\alpha\beta}, \qquad M^{13\alpha} = M^{3\alpha} = 0. \tag{7.10}$$

However, in the case of linear plate theory it is found that the first result holds and only $(7.10)_2$ is not satisfied. The set of equations (7.4)–(7.8) is formally the same as those derived by Green, Naghdi and Wainwright [2] from separate postulates, apart from the extra generality here in the temperature terms.

8. LINEAR THEORY

Here we summarize some aspects of the linear theory of thermoelastic shells based on the results of Section 7. Apart from the extra generality here concerning temperature, the results are expressed in the form given recently by Green and Naghdi [9].

The curvilinear coordinate ξ in Section 3 is chosen so that initially, in (3.6),

$$\mathbf{d} = \mathbf{d}_1 = \mathbf{A}_3, \qquad \mathbf{d}_N = 0 \qquad (N \geq 2), \tag{8.1}$$

where $\mathbf{A}_3$ is the unit normal to the initial middle surface $\xi = 0$ of the shell. The initial values of $\mathbf{a}_i$, $\mathbf{a}^i$, $a_{\alpha\beta}$, $b_{\alpha\beta}, \ldots, d_i$, $\lambda_{i\alpha}$ are denoted by

$$\mathbf{A}_i, \mathbf{A}^i, A_{\alpha\beta}, B_{\alpha\beta}, \ldots, D_i, \Lambda_{i\alpha}. \tag{8.2}$$

We assume that initially the shell is in equilibrium under zero stresses and at uniform temperature θ_0 and we suppose that T^* in (3.7) denotes temperature difference from θ_0. Also

$$D_\alpha = 0, \qquad D_3 = 1, \qquad \Lambda_{\alpha\beta} = -B_{\alpha\beta}, \qquad \Lambda_{3\alpha} = 0. \tag{8.3}$$

If $\mathbf{u}$ is the displacement vector for points on the middle surface of the shell, we set

$$\mathbf{u} = u_i\mathbf{A}^i = u^i\mathbf{A}_i, \quad \mathbf{d} = \mathbf{A}_3 + \boldsymbol{\delta}^*, \quad \boldsymbol{\delta}^* = \delta^{*i}\mathbf{A}_i = \delta^*_i\mathbf{A}^i, \quad d_\alpha = \delta_\alpha, \quad d_3 = 1+\delta_3, \tag{8.4}$$

so that

$$\delta_\alpha = \delta^*_\alpha + u_{3,\alpha} + B^\gamma_\alpha u_\gamma, \qquad \delta_3 = \delta^*_3. \tag{8.5}$$

Also

$$\lambda_{\alpha\beta} - \Lambda_{\alpha\beta} = \rho_{\alpha\beta} - B_{\alpha\beta}\delta_3, \qquad \lambda_{3\alpha} - \Lambda_{3\alpha} = \rho_{3\alpha} + B^\gamma_\alpha\delta_\gamma, \tag{8.6}$$

where

$$-\rho_{\alpha\beta} = u_{3|\alpha\beta} + B^\gamma_{\alpha|\beta}u_\gamma + B^\gamma_\alpha u_{\gamma|\beta} + B^\gamma_\beta u_{\gamma|\alpha} - B_{\alpha\gamma}B^\gamma_\beta u_3 - \delta_{\alpha|\beta}, \qquad \rho_{3\alpha} = \delta_{3,\alpha}. \tag{8.7}$$

The equations of motion (7.4)–(7.6) reduce to†

$$N^{\beta\alpha}{}_{|\alpha} - B^\beta_\alpha V^\alpha + \rho\bar{F}^\beta = 0, \qquad V^\alpha{}_{|\alpha} + B_{\alpha\beta}N^{\beta\alpha} + \rho\bar{F}^3 = 0, \tag{8.8}$$

$$M^{\beta\alpha}{}_{|\alpha} + \rho\bar{L}^\beta = V^\beta, \qquad M^{3\alpha}{}_{|\alpha} + \rho\bar{L}^3 = V^3, \tag{8.9}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} + B^\beta_\lambda M^{\alpha\lambda}, \tag{8.10}$$

† See Green and Naghdi [9]. The notation of the present paper differs from that used by Green and Naghdi [8, 9] in the ordering of some indices.

where ρ is density of the initial shell and $\bar{F}^\beta$, $\bar{F}^3$, $\bar{L}^\beta$, $\bar{L}^3$ are components referred to the initial base vectors $\mathbf{A}_i$. The constitutive relations (7.8) reduce to

$$\begin{aligned} S^N &= -\frac{\partial A}{\partial T_N}, \\ N'^{\alpha\beta} &= \frac{1}{2}\rho\left(\frac{\partial A}{\partial e_{\alpha\beta}}+\frac{\partial A}{\partial e_{\beta\alpha}}\right), \\ M^{i\alpha} &= \rho\frac{\partial A}{\partial \rho_{i\alpha}}, \\ V^i &= \rho\frac{\partial A}{\partial \delta_i}, \end{aligned} \tag{8.11}$$

where S^N denotes entropy difference from initial values and

$$A = A(e_{\alpha\beta}, \rho_{i\alpha}, \delta_i, T_N), \tag{8.12}$$

dependence on $A_{\alpha\beta}$ and $B_{\alpha\beta}$ being understood. The energy equations (7.9) reduce to

$$\rho(r^n+R^n)-\rho\theta_0\dot{S}^n-q^{n\alpha}{}_{|\alpha} = 0. \tag{8.13}$$

9. LINEAR THEORY FOR PLATES

We now consider the results of Section 8 in more detail for the linear theory of plates of uniform thickness h_0. The function (8.12) for a plate which is initially unstressed and at uniform temperature θ_0 is quadratic in the variables $e_{\alpha\beta}$, $\rho_{i\alpha}$, δ_i, T_N. Green and Naghdi [7] have given an explicit expression for an isotropic Cosserat plate when $T_N = 0 (N \geq 1)$. These authors [7] also considered the further restriction such that the strain energy A imitates the symmetry properties of a three-dimensional plate which is transversely isotropic with respect to normals to the plate. In the present context, this latter restriction imposes the condition that A is invariant under the transformations

$$\begin{gathered} u_\alpha \to u_\alpha, \qquad u_3 \to -u_3, \qquad \delta_\alpha \to -\delta_\alpha, \qquad \delta_3 \to \delta_3, \\ e_{\alpha\beta} \to e_{\alpha\beta}, \qquad \rho_{\alpha\beta} \to -\rho_{\alpha\beta}, \qquad \rho_{3\alpha} \to \rho_{3\alpha}, \\ T_{2N} \to T_{2N}, \qquad T_{2N+1} \to -T_{2N+1}, \end{gathered} \tag{9.1}$$

where, for a plate,

$$\rho_{\alpha\beta} = -u_{3|\alpha\beta}+\delta_{\alpha|\beta}. \tag{9.2}$$

Thus

$$\begin{aligned} 2\rho A = {} & [\alpha_1 A^{\alpha\beta}A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e_{\gamma\delta}+\alpha_3 A^{\alpha\beta}\delta_\alpha\delta_\beta+\alpha_4\delta_3^2 \\ & +[\alpha_5 A^{\alpha\beta}A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma}A^{\beta\delta}+\alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\alpha\beta}\rho_{\gamma\delta}+\alpha_8 A^{\alpha\beta}\rho_{3\alpha}\rho_{3\beta}+2\alpha_9 A^{\alpha\beta}e_{\alpha\beta}\delta_3 \\ & +2\delta_3\sum_{N=0}^{\infty}\beta_{2N}T_{2N}+2A^{\alpha\beta}e_{\alpha\beta}\sum_{N=0}^{\infty}\beta'_{2N}T_{2N}+2A^{\alpha\beta}\rho_{\alpha\beta}\sum_{N=0}^{\infty}\beta_{2N+1}T_{2N+1} \\ & +\sum_{M,N=0}^{\infty}\gamma_{2M,2N}T_{2M}T_{2N}+\sum_{M,N=0}^{\infty}\gamma_{2M+1,2N+1}T_{2M+1}T_{2N+1}. \end{aligned} \tag{9.3}$$

From (8.11) we then obtain expressions for stresses and entropy. Inspection of these expressions and of equations (8.8) and (8.9) shows that the basic equations for stretching and bending of the plate separate into two groups. The first group concerned with stretching are

$$\begin{gathered}
N'^{\alpha\beta} = N^{\alpha\beta} = [\alpha_1 A^{\alpha\beta}A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma}A^{\beta\delta}+A^{\alpha\delta}A^{\beta\gamma})]e_{\gamma\delta}+\alpha_9 A^{\alpha\beta}\delta_3+A^{\alpha\beta}\sum_{N=0}^{\infty}\beta'_{2N}T_{2N},\\
M^{3\alpha} = \alpha_8 A^{\alpha\beta}\rho_{3\beta}, \qquad V^3 = \alpha_4\delta_3+\alpha_9 A^{\alpha\beta}e_{\alpha\beta}+\sum_{N=0}^{\infty}\beta_{2N}T_{2N}, \qquad N^{\beta\alpha}{}_{|\alpha}+\rho\bar{F}^{\beta} = 0,\\
M^{3\alpha}{}_{|\alpha}+\rho\bar{L}^3 = V^3,\\
-\rho S^{2N} = \beta_{2N}\delta_3+\beta'_{2N}A^{\alpha\beta}e_{\alpha\beta}+\sum_{M=0}^{\infty}\gamma_{2M,2N}T_{2M},\\
\rho(r^{2n}+R^{2n})-\rho\theta_0\dot{S}^{2n}-q^{2n\alpha}{}_{|\alpha} = 0.
\end{gathered}\tag{9.4}$$

The equations for bending of the plate are

$$\begin{gathered}
M^{\alpha\beta} = [\alpha_5 A^{\alpha\beta}A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma}A^{\beta\delta}+\alpha_7 A^{\alpha\delta}A^{\beta\gamma}]\rho_{\gamma\delta}+A^{\alpha\beta}\sum_{N=0}^{\infty}\beta_{2N+1}T_{2N+1},\\
V^{\alpha} = \alpha_3 A^{\alpha\beta}\delta_{\beta}, \qquad M^{\beta\alpha}{}_{|\alpha}+\rho\bar{L}^{\beta} = V^{\beta}, \qquad V^{\alpha}{}_{|\alpha}+\rho\bar{F}^3 = 0,\\
-\rho S^{2N+1} = \beta_{2N+1}A^{\alpha\beta}\rho_{\alpha\beta}+\sum_{M=0}^{\infty}\gamma_{2M+1,2N+1}T_{2M+1},\\
\rho(r^{2n+1}+R^{2n+1})-\rho\theta_0\dot{S}^{2n+1}-q^{2n+1\alpha}{}_{|\alpha} = 0.
\end{gathered}\tag{9.5}$$

To complete the systems of equations (9.4) and (9.5) we require constitutive equations for R^n and $q^{n\alpha}$ and values for r^n. These can be obtained by separate postulates but we derive them here from the three-dimensional form for the heat conduction vector using (3.29), (3.32) and (3.33) and appropriate conditions at the surfaces $\xi = \pm\frac{1}{2}h_0$ of the plate, where h_0 is a constant. We assume that the temperature of the medium on either side of the plate is given by

$$T_+(\xi > \tfrac{1}{2}h_0), \qquad T_-(\xi < -\tfrac{1}{2}h_0), \tag{9.6}$$

and that at the surfaces of the plate

$$\begin{aligned}
h^* &= H(T^*-T_+) \quad \text{at } \xi = \tfrac{1}{2}h_0,\\
h^* &= H(T^*-T_-) \quad \text{at } \xi = -\tfrac{1}{2}h_0,
\end{aligned}\tag{9.7}$$

where H is a constant. For a transversely isotropic plate

$$q^{*\alpha} = -\kappa A^{\alpha\beta}T^*_{,\beta}, \qquad q^{*3} = -\kappa' T^*_{,3}, \tag{9.8}$$

where

$$\kappa \geq 0, \qquad \kappa' \geq 0. \tag{9.9}$$

It follows from (3.29) and (3.32) that

$$\rho(r^n+R^n) = \rho\bar{r}^n - H[\xi^n(T^*-T_+)]_{\frac{1}{2}h_0} - H[\xi^n(T-T_-)]_{-\frac{1}{2}h_0} - n\kappa'\int_{-\frac{1}{2}h_0}^{\frac{1}{2}h_0}\sum_{N=0}^{\infty}NT_N\xi^{N+n-2}\,d\xi, \tag{9.10}$$

where

$$\rho\bar{r}^n A^{\frac{1}{2}} = k\int_{-\frac{1}{2}h_0}^{\frac{1}{2}h_0} r^*\xi^n\, d\xi. \tag{9.11}$$

Hence

$$\begin{aligned}\rho(r^{2n}+R^{2n}) = \rho\bar{r}^{2n} - H\left[2\sum_{N=0}^{\infty}\left(\frac{h_0}{2}\right)^{2n+2N} T_{2N} - \left(\frac{h_0}{2}\right)^{2n}(T_+ + T_-)\right] \\ -8n\kappa'\sum_{N=0}^{\infty}\frac{NT_{2N}}{2n+2N-1}\left(\frac{h_0}{2}\right)^{2N+2n-1},\end{aligned} \tag{9.12}$$

and

$$\begin{aligned}\rho(r^{2n+1}+R^{2n+1}) = \rho\bar{r}^{2n+1} - H\left[2\sum_{N=0}^{\infty}\left(\frac{h_0}{2}\right)^{2n+2N+2} T_{2N+1} - \left(\frac{h_0}{2}\right)^{2n+1}(T_+ - T_-)\right] \\ -2(2n+1)\kappa'\sum_{N=0}^{\infty}\frac{(2N+1)T_{2N+1}}{2n+2N+1}\left(\frac{h_0}{2}\right)^{2n+2N+1},\end{aligned} \tag{9.13}$$

where, in (9.12) and (9.13), $n = 0, 1, 2, \ldots$.

Expressions for $q^{n\alpha}$ are obtained from (3.33), (9.8) and (3.7). Thus

$$\begin{aligned} q^{2n\alpha} &= -2\kappa A^{\alpha\beta}\sum_{N=0}^{\infty}\frac{T_{2N,\beta}}{2n+2N+1}\left(\frac{h_0}{2}\right)^{2n+2N+1}, \\ q^{2n+1\alpha} &= -2\kappa A^{\alpha\beta}\sum_{N=0}^{\infty}\frac{T_{2N+1,\beta}}{2n+2N+3}\left(\frac{h_0}{2}\right)^{2n+2N+3}. \end{aligned} \tag{9.14}$$

This completes the basic set of equations which, however, involve an infinite set of temperature variables $T_0, T_1, \ldots$. For applications it is necessary to make approximations. Here we restrict our attention to an approximation in which we set

$$T_{2N} = 0, \qquad T_{2N+1} = 0 \qquad (N \geq 1). \tag{9.15}$$

From (9.12) and (9.13) we then have

$$\rho(r^0+R^0) = \rho\bar{r}^0 - H[2T_0 - T_+ - T_-], \tag{9.16}$$

$$\rho(r^1+R^1) = \rho\bar{r}^1 - H[\tfrac{1}{2}h_0^2 T_1 - \tfrac{1}{2}h_0(T_+ - T_-)] - \kappa' h_0 T_1. \tag{9.17}$$

Also, equations (9.14) reduce to

$$q^{0\alpha} = -\kappa h_0 A^{\alpha\beta} T_{0,\beta}, \tag{9.18}$$

$$q^{1\alpha} = -\frac{\kappa h_0^3}{12} A^{\alpha\beta} T_{1,\beta}. \tag{9.19}$$

The energy equations in (9.4) and (9.5) corresponding to $n = 0$ now become

$$\theta_0\frac{\partial}{\partial t}(\gamma_{0,0}T_0 + \beta_0\delta_3 + \beta_0' A^{\alpha\beta}e_{\alpha\beta}) + \kappa h_0\nabla^2 T_0 + \rho\bar{r}^0 - H[2T_0 - T_+ - T_-] = 0, \tag{9.20}$$

and

$$\theta_0 \frac{\partial}{\partial t}(\gamma_{1,1} T_1 + \beta_1 A^{\alpha\beta} \rho_{\alpha\beta}) + \frac{\kappa h_0^3}{12} \nabla^2 T_1 + \rho \bar{r}^1 - \tfrac{1}{2} H[h_0^2 T_1 - h_0(T_+ - T_-)] - \kappa' h_0 T_1 = 0, \quad (9.21)$$

where ∇^2 is the two dimensional Laplacian operator. If in (9.20) and (9.21) the strain components (which represent the effect of thermo-mechanical coupling) are omitted, we recover equations of the same form as those derived by Bolotin [12] for shells.

We close this Section with some remarks about the determination of the constants in (9.3). By comparing some exact solutions from the three dimensional linear elasticity with corresponding solutions predicted by the approximate theory, we can identify most of the elastic coefficients which occur in the approximate value of A in (9.3). In this way, some of the coefficients $\alpha_1, \ldots, \alpha_9$ were determined previously [9] as follows:

$$\begin{aligned} &\alpha_1 = \alpha_9 = \frac{\eta(1-\eta)D}{1-2\eta}, \qquad \alpha_2 = \tfrac{1}{2}(1-\eta)D, \qquad \alpha_4 = \frac{(1-\eta)^2 D}{1-2\eta}, \qquad \alpha_5 = \eta B, \\ &\alpha_6 = \alpha_7 = \tfrac{1}{2}(1-\eta)B, \end{aligned} \quad (9.22)$$

where

$$D = \frac{Eh_0}{1-\eta^2}, \qquad B = \frac{Eh_0^3}{12(1-\eta^2)}, \quad (9.23)$$

E being Young's modulus and η Poisson's ratio. If we include $(7.10)_2$ in the present approximation, we find that $\alpha_8 = 0$; if we disregard $(7.10)_2$, then α_8 is arbitrary†. The coefficient α_3 cannot be determined as a constant by comparison with a three dimensional solution, and it seems preferable to allow α_3 to have different possible values depending on the particular context in which the approximate theory is used. In particular, consider the problem of torsion of a rectangular strip of breadth a and thickness h_0 and use (9.5) to obtain the formula for torsional rigidity, i.e.

$$\mu \frac{h_0^3 a}{3}\left[1 - \frac{2\lambda}{a} \tanh \frac{a}{2\lambda}\right], \quad (9.24)$$

where $\mu = E/[2(1+\eta)]$ and

$$\lambda^2 = \frac{\alpha_6}{\alpha_3}. \quad (9.25)$$

It is known from a similar result in Reissner's plate theory [14] that the formula (9.24) gives good results for a wide range of values of a/h_0 if

$$\alpha_3 = \tfrac{5}{6}\mu h_0. \quad (9.26)$$

In view of (3.7) and the approximation (9.15), it is not difficult to make the identification

$$\beta_0 = \beta_0' = -\frac{Eh_0}{1-2\eta}\alpha, \qquad \beta_1 = -\frac{Eh_0^3}{12(1-\eta)}\alpha, \quad (9.27)$$

where α is the coefficient of linear expansion.

† We recall that in the theory of a Cosserat plate [7], the coefficient corresponding to α_8 is not zero.

10. RODS

The notation in this and following sections must be regarded as separate from that in Sections 3 to 9 so that no confusion need arise over different uses of the same symbol.

The parametric equations

$$\theta^\alpha = 0 \tag{10.1}$$

define a curve c in space at time t. We assume that the curve c is sufficiently smooth and nonintersecting. The position vector $\mathbf{r}$ of any point of c is given by

$$\mathbf{r} = \mathbf{r}(\theta, t) = \mathbf{r}^*(0, 0, \theta^3, t) \qquad (\theta^3 \equiv \theta). \tag{10.2}$$

We also assume that the region of space occupied by the continuum is some neighborhood of c which is bounded by a surface

$$f(\theta^1, \theta^2) = 0, \tag{10.3}$$

such that $\theta =$ constant are curved sections of this continuum bounded by closed curves. We call such a continuum a rod. We fix the relation of the curve c to the boundary surface (10.3) by imposing the conditions

$$\int\int \rho^*(g)^{\frac{1}{2}}\theta^\alpha \, d\theta^1 \, d\theta^2 = \int\int k(\theta^1, \theta^2, \theta)\theta^\alpha \, d\theta^1 \, d\theta^2 = 0, \tag{10.4}$$

the integration being over any surface $\theta =$ constant bounded by (10.3). We observe that these conditions are independent of time t so that once c is determined by such equations (in, say, a reference state) it is determined for all time.

We assume that the position vector and temperature of any point of the rod at time t can be represented by the expansions

$$\mathbf{r}^* = \mathbf{r}(\theta, t) + \sum_N \theta^{\alpha_1}\theta^{\alpha_2} \ldots \theta^{\alpha_N}\mathbf{d}_{\alpha_1\alpha_2\cdots\alpha_N} \tag{10.5}$$

$$T^* = T_0(\theta, t) + \sum_N \theta^{\alpha_1}\theta^{\alpha_2} \ldots \theta^{\alpha_N} T_{\alpha_1\alpha_2\cdots\alpha_N} \tag{10.6}$$

where $\mathbf{d}_{\alpha_1\alpha_2\cdots\alpha_N}$ are vector functions and $T_{\alpha_1\alpha_2\cdots\alpha_N}$ are scalar functions of θ, t, both being completely symmetric in the indices $\alpha_1, \alpha_2, \ldots, \alpha_N$. The summation in (10.5) and (10.6) is over all values of $\alpha_1, \alpha_2, \ldots, \alpha_N = 1, 2$ and $N = 1, 2, 3, \ldots$. We assume that (10.5) and (10.6) may be differentiated as many times as required with respect to any of their variables.

From (2.2) and (10.5) we have

$$\mathbf{v}^* = \mathbf{v} + \sum_N \theta^{\alpha_1}\theta^{\alpha_2} \ldots \theta^{\alpha_N}\mathbf{w}_{\alpha_1\alpha_2\cdots\alpha_N} \tag{10.7}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{w}_{\alpha_1\alpha_2\cdots\alpha_N} = \dot{\mathbf{d}}_{\alpha_1\alpha_2\cdots\alpha_N}. \tag{10.8}$$

For later convenience we put

$$\mathbf{d}_\alpha = \mathbf{a}_\alpha, \qquad \mathbf{w}_\alpha = \dot{\mathbf{a}}_\alpha. \tag{10.9}$$

We call $\mathbf{d}_{\alpha_1\alpha_2\cdots\alpha_N}$ directors and $\mathbf{w}_{\alpha_1\alpha_2\cdots\alpha_N}$ director velocities and observe that $\mathbf{d}_{\alpha_1\alpha_2\cdots\alpha_N}$ are unchanged in length when the rod is subjected to superposed rigid body motions. We also use the notation

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \theta}, \qquad a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \tag{10.10}$$

where $\mathbf{a}_3$ is a tangent vector to the curve (10.2). From (10.5) and (2.1) we see that

$$\begin{aligned} \mathbf{g}_\beta &= \mathbf{a}_\beta + \sum_{N=2}^{\infty} N\theta^{\alpha_2}\theta^{\alpha_3}\ldots\theta^{\alpha_N}\mathbf{d}_{\beta\alpha_2\alpha_3\cdots\alpha_N}, \\ \mathbf{g}_3 &= \mathbf{a}_3 + \sum_{N=1}^{\infty} \theta^{\alpha_1}\theta^{\alpha_2}\ldots\theta^{\alpha_N}\partial\mathbf{d}_{\alpha_1\alpha_2\cdots\alpha_N}/\partial\theta. \end{aligned} \tag{10.11}$$

With the displacement function (10.5) the restriction analogous to (3.14) is

$$[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] > 0. \tag{10.12}$$

For the rest of this Section we consider an arbitrary element of the rod bounded by the surfaces $\theta = \alpha$, $\theta = \beta$ ($\beta \geq \theta \geq \alpha$), and a surface (10.3).

With the help of (10.7), the energy equation (2.3) was reduced to one dimensional form in† I. We quote the final result and refer readers to the previous paper for details. Thus

$$\begin{aligned} &\frac{D}{Dt}\int_{\phi_1}^{\phi_2} \rho\left[U + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \sum_{N=2}^{\infty} k^{\alpha_1\cdots\alpha_N}\mathbf{v}\cdot\mathbf{w}_{\alpha_1\cdots\alpha_N} + \frac{1}{2}\sum_{M,N=1}^{\infty} k^{\alpha_1\cdots\alpha_N\beta_1\cdots\beta_M}\mathbf{w}_{\alpha_1\cdots\alpha_N}\cdot\mathbf{w}_{\beta_1\cdots\beta_M}\right](a_{33})^{\frac{1}{2}}\,d\theta \\ &= \int_{\phi_1}^{\phi_2} \rho\left[r + \mathbf{f}\cdot\mathbf{v} + \sum_{N=1}^{\infty} \mathbf{l}^{\alpha_1\cdots\alpha_N}\cdot\mathbf{w}_{\alpha_1\cdots\alpha_N}\right](a_{33})^{\frac{1}{2}}\,d\theta + \left[\mathbf{n}\cdot\mathbf{v} + \sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_N}\cdot\mathbf{w}_{\alpha_1\cdots\alpha_N} - h\right]_{\phi_1}^{\phi_2}, \end{aligned} \tag{10.13}$$

where $\alpha \leq \phi_1 \leq \theta \leq \phi_2 \leq \beta$ and

$$[\psi(\theta, t)]_{\phi_1}^{\phi_2} = \psi(\phi_2, t) - \psi(\phi_1, t).$$

Also‡

$$\begin{aligned} \rho(a_{33})^{\frac{1}{2}} &= \int\int \rho^*(g)^{\frac{1}{2}}\,d\theta^1\,d\theta^2 = \int\int k\,d\theta^1\,d\theta^2, \\ \rho k^{\alpha_1\cdots\alpha_N}(a_{33})^{\frac{1}{2}} &= \int\int k\theta^{\alpha_1}\ldots\theta^{\alpha_N}\,d\theta^1\,d\theta^2 \qquad (N \geq 2), \end{aligned} \tag{10.14}$$

$$\rho r(a_{33})^{\frac{1}{2}} = \int\int kr^*\,d\theta^1\,d\theta^2 - \oint h^*(u^1\,d\theta^2 - u^2\,d\theta^1)g^{\frac{1}{2}}, \tag{10.15}$$

$$\rho\mathbf{f}(a_{33})^{\frac{1}{2}} = \int\int k\mathbf{f}^*\,d\theta^1\,d\theta^2 + \oint(\mathbf{T}_1\,d\theta^2 - \mathbf{T}_2\,d\theta^1), \tag{10.16}$$

$$\rho\mathbf{l}^{\alpha_1\cdots\alpha_N}(a_{33})^{\frac{1}{2}} = \int\int k\mathbf{f}^*\theta^{\alpha_1}\ldots\theta^{\alpha_N}\,d\theta^1\,d\theta^2 + \oint\theta^{\alpha_1}\ldots\theta^{\alpha_N}(\mathbf{T}_1\,d\theta^2 - \mathbf{T}_2\,d\theta^1), \tag{10.17}$$

$$\rho U(a_{33})^{\frac{1}{2}} = \int\int kU^*\,d\theta^1\,d\theta^2. \tag{10.18}$$

† As noted in section 1, I refers to Ref. [1].
‡ In view of (10.4), $k^\alpha = 0$.

In (10.15)–(10.18), the double integrals are over any section θ = const. of the rod bounded by the surface (10.3) and the line integrals are along the curve

$$\theta = \text{constant}, \qquad f(\theta^1, \theta^2) = 0. \tag{10.19}$$

In addition,

$$h = \int\int h^*(gg^{33})^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 = \int\int q^{*3}g^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.20}$$

$$\mathbf{n} = \int\int \mathbf{T}_3\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.21}$$

$$\mathbf{p}^{\alpha_1\cdots\alpha_N} = \int\int \theta^{\alpha_1}\cdots\theta^{\alpha_N}\mathbf{T}_3\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2. \tag{10.22}$$

We now substitute

$$\phi = \theta^{\alpha_1}\ldots\theta^{\alpha_n}$$

in (2.15) and follow a procedure similar to that used in obtaining (10.13) from (2.3). Thus

$$\begin{aligned}
&\frac{D}{Dt}\int_{\phi_1}^{\phi_2}\rho\Bigg[U^{\alpha_1\cdots\alpha_n}+\frac{1}{2}k^{\alpha_1\cdots\alpha_n}\mathbf{v}\cdot\mathbf{v}+\sum_{N=1}^{\infty}k^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}\mathbf{v}\cdot\mathbf{w}_{\beta_1\cdots\beta_N}\\
&\qquad+\frac{1}{2}\sum_{M,N=1}^{\infty}k^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N\gamma_1\cdots\gamma_M}\mathbf{w}_{\beta_1\cdots\beta_N}\cdot\mathbf{w}_{\gamma_1\cdots\gamma_M}\Bigg](a_{33})^{\frac{1}{2}}\,\mathrm{d}\theta\\
&=\int_{\phi_1}^{\phi_2}\Bigg\{\rho\Bigg[r^{\alpha_1\cdots\alpha_n}+R^{\alpha_1\cdots\alpha_n}+\mathbf{l}^{\alpha_1\cdots\alpha_n}\cdot\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{l}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}\cdot\mathbf{w}_{\beta_1\cdots\beta_N}\Bigg](a_{33})^{\frac{1}{2}}\\
&\qquad-\boldsymbol{\pi}^{\alpha_1\cdots\alpha_n}\cdot\mathbf{v}-\sum_{N=1}^{\infty}\big(\boldsymbol{\omega}^{\alpha_1(\alpha_2\cdots\alpha_n\beta_1\cdots\beta_N)}+\ldots+\boldsymbol{\omega}^{\alpha_n(\alpha_1\cdots\alpha_{n-1}\beta_1\cdots\beta_N)}\big)\cdot\mathbf{w}_{\beta_1\cdots\beta_N}\Bigg\}\,\mathrm{d}\theta\\
&\qquad+\Bigg[\mathbf{p}^{\alpha_1\cdots\alpha_n}\cdot\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}\cdot\mathbf{w}_{\beta_1\cdots\beta_N}-h^{\alpha_1\cdots\alpha_n}\Bigg]_{\phi_1}^{\phi_2}
\end{aligned}\tag{10.23}$$

where, in addition to quantities already defined, we have

$$\boldsymbol{\omega}^{\alpha_1\cdots\alpha_n} = \boldsymbol{\omega}^{\alpha_1(\alpha_2\cdots\alpha_n)} = \int\int \mathbf{T}_{\alpha_1}\theta^{\alpha_2}\ldots\theta^{\alpha_n}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 \tag{10.24}$$

$$\boldsymbol{\pi}^{\alpha_1\cdots\alpha_n} = \boldsymbol{\omega}^{\alpha_1(\alpha_2\cdots\alpha_n)}+\boldsymbol{\omega}^{\alpha_2(\alpha_1\cdots\alpha_n)}+\ldots+\boldsymbol{\omega}^{\alpha_n(\alpha_1\cdots\alpha_{n-1})} \tag{10.25}$$

$$\boldsymbol{\pi}^{\alpha} = \boldsymbol{\omega}^{\alpha} = \int\int \mathbf{T}_{\alpha}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.26}$$

$$h^{\alpha_1\cdots\alpha_n} = \int\int \theta^{\alpha_1}\ldots\theta^{\alpha_n}h^*(gg^{33})^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 = \int\int \theta^{\alpha_1}\ldots\theta^{\alpha_n}q^{*3}g^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.27}$$

$$\rho R^{\alpha_1\cdots\alpha_n}(a_{33})^{\frac{1}{2}} = \int\int [q^{*\alpha_1}\theta^{\alpha_2}\ldots\theta^{\alpha_n}+\ldots+q^{*\alpha_n}\theta^{\alpha_1}\ldots\theta^{\alpha_{n-1}}]g^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.28}$$

$$\rho R^{\alpha}(a_{33})^{\frac{1}{2}} = \int\int q^{*\alpha}g^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.29}$$

$$\rho U^{\alpha_1\cdots\alpha_n}(a_{33})^{\frac{1}{2}} = \int\int k\theta^{\alpha_1}\ldots\theta^{\alpha_n}U^*\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \tag{10.30}$$

$$\rho r^{\alpha_1\cdots\alpha_n}(a_{33})^{\frac{1}{2}} = \int\int k\theta^{\alpha_1}\ldots\theta^{\alpha_n}r^*\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 - \oint\theta^{\alpha_1}\ldots\theta^{\alpha_n}h^*(u^1\,\mathrm{d}\theta^2-u^2\,\mathrm{d}\theta^1)g^{\frac{1}{2}}. \tag{10.31}$$

11. FURTHER DEVELOPMENT OF ENERGY EQUATION FOR RODS

In view of $(10.14)_1$, $\rho(a_{33})^{\frac{1}{2}}$ is a function of θ and is independent of time so we set

$$\rho(a_{33})^{\frac{1}{2}} = \lambda(\theta). \tag{11.1}$$

Equations (10.13) and (10.23) in point form become

$$\begin{aligned}\lambda r-\lambda\dot{U}+\left(\frac{\partial\mathbf{n}}{\partial\theta}+\lambda\bar{\mathbf{f}}\right)\cdot\mathbf{v}+\mathbf{n}\cdot\frac{\partial\mathbf{v}}{\partial\theta}+\sum_{N=1}^{\infty}\left(\lambda\mathbf{q}^{\alpha_1\cdots\alpha_N}+\frac{\partial\mathbf{p}^{\alpha_1\cdots\alpha_N}}{\partial\theta}\right)\cdot\mathbf{w}_{\alpha_1\cdots\alpha_N}&\\+\sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_N}\cdot\frac{\partial\mathbf{w}_{\alpha_1\cdots\alpha_N}}{\partial\theta}-\frac{\partial h}{\partial\theta}=0,&\end{aligned} \tag{11.2}$$

and

$$\begin{aligned}&\lambda(r^{\alpha_1\cdots\alpha_n}+R^{\alpha_1\cdots\alpha_n}-\dot{U}^{\alpha_1\cdots\alpha_n})+\left(\lambda\mathbf{q}^{\alpha_1\cdots\alpha_n}+\frac{\partial\mathbf{p}^{\alpha_1\cdots\alpha_n}}{\partial\theta}-\boldsymbol{\pi}^{\alpha_1\cdots\alpha_n}\right)\cdot\mathbf{v}+\mathbf{p}^{\alpha_1\cdots\alpha_n}\cdot\frac{\partial\mathbf{v}}{\partial\theta}\\&+\sum_{N=1}^{\infty}\left(\lambda\mathbf{q}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}+\frac{\partial\mathbf{p}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}}{\partial\theta}-\boldsymbol{\omega}^{\alpha_1(\alpha_2\cdots\alpha_n\beta_1\cdots\beta_N)}-\ldots-\boldsymbol{\omega}^{\alpha_n(\alpha_1\cdots\alpha_{n-1}\beta_1\cdots\beta_N)}\right)\cdot\mathbf{w}_{\beta_1\cdots\beta_N}\\&+\sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}\cdot\frac{\partial\mathbf{w}_{\beta_1\cdots\beta_N}}{\partial\theta}-\frac{\partial h^{\alpha_1\cdots\alpha_n}}{\partial\theta}=0,\end{aligned} \tag{11.3}$$

where, in addition to notation already specified, we have

$$\begin{aligned}\bar{\mathbf{f}}&=\mathbf{f}-\dot{\mathbf{v}}-\sum_{N=2}^{\infty}k^{\alpha_1\cdots\alpha_N}\dot{\mathbf{w}}_{\alpha_1\cdots\alpha_N},\\\mathbf{q}^{\alpha_1\cdots\alpha_n}&=\mathbf{l}^{\alpha_1\cdots\alpha_n}-k^{\alpha_1\cdots\alpha_n}\dot{\mathbf{v}}-\sum_{N=1}^{\infty}k^{\alpha_1\cdots\alpha_n\gamma_1\cdots\gamma_N}\dot{\mathbf{w}}_{\gamma_1\cdots\gamma_N}.\end{aligned} \tag{11.4}$$

We consider a motion of the rod at time t in which the velocities differ from the given velocities only by superposed uniform translational rigid body velocities and we assume that these do not change the quantities r, $r^{\alpha_1\cdots\alpha_n}$, $R^{\alpha_1\cdots\alpha_n}$, U, $U^{\alpha_1\cdots\alpha_n}$, $\mathbf{n}$, $\mathbf{p}^{\alpha_1\cdots\alpha_n}$, $\mathbf{q}^{\alpha_1\cdots\alpha_n}$, $\bar{\mathbf{f}}$, $\boldsymbol{\omega}^{\alpha_1(\alpha_2\cdots\alpha_n)}$, h, $h^{\alpha_1\cdots\alpha_n}$. It then follows from (11.2) and (11.3) that

$$\frac{\partial\mathbf{n}}{\partial\theta}+\lambda\bar{\mathbf{f}}=0,\qquad\lambda\mathbf{q}^{\alpha_1\cdots\alpha_n}+\frac{\partial\mathbf{p}^{\alpha_1\cdots\alpha_n}}{\partial\theta}=\boldsymbol{\pi}^{\alpha_1\cdots\alpha_n}. \tag{11.5}$$

With the help of (11.5) equations (11.2) and (11.3) reduce to

$$\lambda r-\lambda\dot{U}+\mathbf{n}\cdot\frac{\partial\mathbf{v}}{\partial\theta}+\sum_{N=1}^{\infty}\boldsymbol{\pi}^{\alpha_1\cdots\alpha_N}\cdot\mathbf{w}_{\alpha_1\cdots\alpha_N}+\sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_N}\cdot\frac{\partial\mathbf{w}_{\alpha_1\cdots\alpha_N}}{\partial\theta}-\frac{\partial h}{\partial\theta}=0, \tag{11.6}$$

and

$$\lambda(r^{\alpha_1\cdots\alpha_n}+R^{\alpha_1\cdots\alpha_n}-\dot{U}^{\alpha_1\cdots\alpha_n})+\mathbf{p}^{\alpha_1\cdots\alpha_n}\cdot\frac{\partial\mathbf{v}}{\partial\theta}$$
$$+\sum_{N=1}^{\infty}(\boldsymbol{\omega}^{\beta_1(\alpha_1\cdots\alpha_n\beta_2\cdots\beta_N)}+\ldots+\boldsymbol{\omega}^{\beta_N(\alpha_1\cdots\alpha_n\beta_1\cdots\beta_{N-1})})\cdot\mathbf{w}_{\beta_1\cdots\beta_N} \tag{11.7}$$
$$+\sum_{N=1}^{\infty}\mathbf{p}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}\cdot\frac{\partial\mathbf{w}_{\beta_1\cdots\beta_N}}{\partial\theta}-\frac{\partial h^{\alpha_1\cdots\alpha_n}}{\partial\theta}=0.$$

Next we consider a motion of the rod in which the velocities differ from the given velocities only by a superposed uniform rigid body angular velocity, the rod having the same orientation as before. Then, assuming that the same quantities mentioned above are unaltered by such rigid body motions, from (11.6) and (11.7), we deduce the equations

$$\mathbf{a}_3\times\mathbf{n}+\sum_{N=1}^{\infty}\mathbf{d}_{\alpha_1\cdots\alpha_N}\times\boldsymbol{\pi}^{\alpha_1\cdots\alpha_N}+\sum_{N=1}^{\infty}\frac{\partial\mathbf{d}_{\alpha_1\cdots\alpha_N}}{\partial\theta}\times\mathbf{p}^{\alpha_1\cdots\alpha_N}=0, \tag{11.8}$$

and

$$\mathbf{a}_3\times\mathbf{p}^{\alpha_1\cdots\alpha_n}+\mathbf{a}_\alpha\times\boldsymbol{\omega}^{\alpha(\alpha_1\cdots\alpha_n)}+\sum_{N=2}^{\infty}\mathbf{d}_{\beta_1\cdots\beta_N}\times(\boldsymbol{\omega}^{\beta_1(\alpha_1\cdots\alpha_n\beta_2\cdots\beta_N)}+\ldots+\boldsymbol{\omega}^{\beta_N(\alpha_1\cdots\alpha_n\beta_1\cdots\beta_{N-1})})$$
$$+\sum_{N=1}^{\infty}\frac{\partial\mathbf{d}_{\beta_1\cdots\beta_N}}{\partial\theta}\times\mathbf{p}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N}=0. \tag{11.9}$$

It is of interest to note that the equations of motion (11.5) can be obtained directly from (2.5) and (10.7); and the equations (11.8) and (11.9) can be deduced from (2.6) and (10.7).

With the help of (11.8) and (11.9), the equations (11.6) and (11.7) can be simplified. For this purpose we need to introduce further notation. Thus, let

$$\mathbf{a}^i\cdot\mathbf{a}_j=\delta^i_j,\qquad a^{ij}=\mathbf{a}^i\cdot\mathbf{a}^j,\qquad \frac{\partial\mathbf{a}_i}{\partial\theta}=\kappa_{is}\mathbf{a}^s=\kappa_{i\cdot}^{\ s}\mathbf{a}_s, \tag{11.10}$$

and if $\mathbf{b}$ is a vector such that

$$\mathbf{b}=b^i\mathbf{a}_i=b_i\mathbf{a}^i, \tag{11.11}$$

then

$$\frac{\partial\mathbf{b}}{\partial\theta}=\frac{\delta b^i}{\delta\theta}\mathbf{a}_i=\frac{\delta b_i}{\delta\theta}\mathbf{a}^i, \tag{11.12}$$

where

$$\frac{\delta b^i}{\delta\theta}=\frac{\partial b^i}{\partial\theta}+\kappa_{r\cdot}^{\ i}b^r,\qquad \frac{\delta b_i}{\delta\theta}=\frac{\partial b_i}{\partial\theta}-\kappa_{i\cdot}^{\ r}b_r. \tag{11.13}$$

Next

$$\begin{aligned}\mathbf{d}_{\alpha_1\cdots\alpha_N}&=d_{\alpha_1\cdots\alpha_N i}\mathbf{a}^i=d_{\alpha_1\cdots\alpha_N\cdot}^{\ \ i}\mathbf{a}_i,\\ \frac{\partial\mathbf{d}_{\alpha_1\cdots\alpha_N}}{\partial\theta}&=\lambda_{\alpha_1\cdots\alpha_N i}\mathbf{a}^i=\lambda_{\alpha_1\cdots\alpha_N\cdot}^{\ \ i}\mathbf{a}_i,\\ \lambda_{\alpha_1\cdots\alpha_N i}&=\frac{\delta d_{\alpha_1\cdots\alpha_N i}}{\delta\theta}.\end{aligned} \tag{11.14}$$

and

$$\mathrm{d}_{\alpha i} = a_{\alpha i}, \qquad \lambda_{\alpha i} = \kappa_{\alpha i}, \tag{11.15}$$

in view of (10.9). Also†

$$\bar{\mathbf{f}} = \bar{f}^i \mathbf{a}_i, \qquad \mathbf{q}^{\alpha_1 \cdots \alpha_N} = q^{\alpha_1 \cdots \alpha_N i} \mathbf{a}_i, \tag{11.16}$$

$$\mathbf{n} = n^i \mathbf{a}_i, \qquad \mathbf{p}^{\alpha_1 \cdots \alpha_N} = p^{\alpha_1 \cdots \alpha_N i} \mathbf{a}_i, \tag{11.17}$$

$$\boldsymbol{\pi}^{\alpha_1 \cdots \alpha_N} = \pi^{\alpha_1 \cdots \alpha_N i} \mathbf{a}_i, \qquad \pi^{\alpha_1 \cdots \alpha_N i} = \frac{\delta p^{\alpha_1 \cdots \alpha_N i}}{\delta \theta} + \lambda q^{\alpha_1 \cdots \alpha_N i}, \tag{11.18}$$

$$\begin{aligned} \boldsymbol{\chi}^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N} &= \boldsymbol{\omega}^{\beta_1(\alpha_1 \cdots \alpha_n \beta_2 \cdots \beta_N)} + \ldots + \boldsymbol{\omega}^{\beta_N(\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_{N-1})}, \\ \boldsymbol{\chi}^{(\alpha_1 \cdots \alpha_n)\beta} &= \boldsymbol{\omega}^{\beta(\alpha_1 \cdots \alpha_n)}, \qquad \boldsymbol{\chi}^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N} = \chi^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N i} \mathbf{a}_i, \\ \boldsymbol{\pi}^{\alpha_1 \cdots \alpha_n} &= \boldsymbol{\chi}^{(\alpha_2 \cdots \alpha_n)\alpha_1} + \ldots + \boldsymbol{\chi}^{(\alpha_1 \cdots \alpha_{n-1})\alpha_n}. \end{aligned} \tag{11.19}$$

Then, using (11.8) and (11.9), equations (11.6) and (11.7) become

$$\lambda r - \lambda \dot{U} + \bar{\pi}^{(\lambda\mu)} \eta_{\lambda\mu} + \bar{\pi}^{\lambda} \eta_{\lambda 3} + \bar{\pi} \eta_{33} + p^{\alpha i} \dot{\kappa}_{\alpha i} + \sum_{N=2}^{\infty} \pi^{\alpha_1 \cdots \alpha_N i} \dot{d}_{\alpha_1 \cdots \alpha_N i} + \sum_{N=2}^{\infty} p^{\alpha_1 \cdots \alpha_N i} \dot{\lambda}_{\alpha_1 \cdots \alpha_N i} - \frac{\partial h}{\partial \theta} = 0, \tag{11.20}$$

and

$$\begin{aligned} &\lambda(r^{\alpha_1 \cdots \alpha_n} + R^{\alpha_1 \cdots \alpha_n} - \dot{U}^{\alpha_1 \cdots \alpha_n}) + \bar{\pi}^{(\lambda\mu)(\alpha_1 \cdots \alpha_n)} \eta_{\lambda\mu} + \bar{\pi}^{\lambda(\alpha_1 \cdots \alpha_n)} \eta_{\lambda 3} + \bar{\pi}^{(\alpha_1 \cdots \alpha_n)} \eta_{33} + p^{\alpha_1 \cdots \alpha_n \beta i} \dot{\kappa}_{\beta i} \\ &\quad + \sum_{N=2}^{\infty} \chi^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N i} \dot{d}_{\beta_1 \cdots \beta_N i} + \sum_{N=2}^{\infty} p^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N i} \dot{\lambda}_{\beta_1 \cdots \beta_N i} - \frac{\partial h^{\alpha_1 \cdots \alpha_n}}{\partial \theta} = 0, \end{aligned} \tag{11.21}$$

where

$$2\eta_{ij} = \dot{a}_{ij}, \tag{11.22}$$

$$\begin{aligned} 2\bar{\pi}^{(\lambda\mu)} = {} & \pi^{\lambda\mu} + \pi^{\mu\lambda} - p^{\alpha\lambda} \kappa_{\alpha\cdot}^{\ \mu} - p^{\alpha\mu} \kappa_{\alpha\cdot}^{\ \lambda} - \sum_{N=2}^{\infty} (\pi^{\alpha_1 \cdots \alpha_N \lambda} d_{\alpha_1 \cdots \alpha_N\cdot}^{\ \mu} + \pi^{\alpha_1 \cdots \alpha_N \mu} d_{\alpha_1 \cdots \alpha_N\cdot}^{\ \lambda}) \\ & - \sum_{N=2}^{\infty} (p^{\alpha_1 \cdots \alpha_N \lambda} \lambda_{\alpha_1 \cdots \alpha_N\cdot}^{\ \mu} + p^{\alpha_1 \cdots \alpha_N \mu} \lambda_{\alpha_1 \cdots \alpha_N\cdot}^{\ \lambda}), \end{aligned} \tag{11.23}$$

$$\begin{aligned} 2\bar{\pi}^{(\lambda\mu)(\alpha_1 \cdots \alpha_n)} = {} & \chi^{(\alpha_1 \cdots \alpha_n)\mu\lambda} + \chi^{(\alpha_1 \cdots \alpha_n)\lambda\mu} - p^{\alpha_1 \cdots \alpha_n \beta \lambda} \kappa_{\beta\cdot}^{\ \mu} - p^{\alpha_1 \cdots \alpha_n \beta \mu} \kappa_{\beta\cdot}^{\ \lambda} \\ & - \sum_{N=2}^{\infty} (\chi^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N \lambda} d_{\beta_1 \cdots \beta_N\cdot}^{\ \mu} + \chi^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N \mu} d_{\beta_1 \cdots \beta_N\cdot}^{\ \lambda}) \\ & - \sum_{N=2}^{\infty} (p^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N \lambda} \lambda_{\beta_1 \cdots \beta_N\cdot}^{\ \mu} + p^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N \mu} \lambda_{\beta_1 \cdots \beta_N\cdot}^{\ \lambda}), \end{aligned} \tag{11.24}$$

$$\bar{\pi}^{\lambda} = 2(n^{\lambda} - p^{\alpha 3} \kappa_{\alpha\cdot}^{\ \lambda}) - 2 \sum_{N=2}^{\infty} \pi^{\alpha_1 \cdots \alpha_N 3} d_{\alpha_1 \cdots \alpha_N\cdot}^{\ \lambda} - 2 \sum_{N=2}^{\infty} p^{\alpha_1 \cdots \alpha_N 3} \lambda_{\alpha_1 \cdots \alpha_N\cdot}^{\ \lambda}, \tag{11.25}$$

† The definition of $\pi^{\alpha_1 \cdots \alpha_N i}$ differs from that used in previous papers.

$$\bar{\bar{\pi}}^{\lambda(\alpha_1\cdots\alpha_n)} = 2(p^{\alpha_1\cdots\alpha_n\lambda} - p^{\alpha_1\cdots\alpha_n\beta 3}\kappa_{\beta\cdot}{}^{\lambda}) - 2\sum_{N=2}^{\infty} \chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N 3} d_{\beta_1\cdots\beta_N\cdot}{}^{\lambda}$$
$$-2\sum_{N=2}^{\infty} p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N 3}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{\lambda}, \tag{11.26}$$

$$\bar{\bar{\pi}} = n^3 - p^{\alpha 3}\kappa_{\alpha\cdot}{}^{3} - \sum_{N=2}^{\infty} \pi^{\alpha_1\cdots\alpha_N 3} d_{\alpha_1\cdots\alpha_N\cdot}{}^{3} - \sum_{N=2}^{n} p^{\alpha_1\cdots\alpha_N 3}\lambda_{\alpha_1\cdots\alpha_N\cdot}{}^{3}, \tag{11.27}$$

$$\bar{\bar{\pi}}^{(\alpha_1\cdots\alpha_n)} = p^{\alpha_1\cdots\alpha_n 3} - p^{\alpha_1\cdots\alpha_n\beta 3}\kappa_{\beta\cdot}{}^{3} - \sum_{N=2}^{\infty} \chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N 3} d_{\beta_1\cdots\beta_N\cdot}{}^{3} - \sum_{N=2}^{\infty} p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N 3}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{3}. \tag{11.28}$$

The results (11.23)–(11.28) are obtained with the help of the component forms of equations (11.8) and (11.9) which are

$$\pi^{\lambda\mu} - \pi^{\mu\lambda} + p^{\alpha\mu}\kappa_{\alpha\cdot}{}^{\lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}{}^{\mu} + \sum_{N=2}^{\infty} (\pi^{\alpha_1\cdots\alpha_N\mu} d_{\alpha_1\cdots\alpha_N\cdot}{}^{\lambda} - \pi^{\alpha_1\cdots\alpha_N\lambda} d_{\alpha_1\cdots\alpha_N\cdot}{}^{\mu})$$
$$+\sum_{N=2}^{\infty} (p^{\alpha_1\cdots\alpha_N\mu}\lambda_{\alpha_1\cdots\alpha_N\cdot}{}^{\lambda} - p^{\alpha_1\cdots\alpha_N\lambda}\lambda_{\alpha_1\cdots\alpha_N\cdot}{}^{\mu}) = 0, \tag{11.29}$$

$$\pi^{\lambda 3} - n^{\lambda} + p^{\alpha 3}\kappa_{\alpha\cdot}{}^{\lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}{}^{3} + \sum_{N=2}^{\infty} (\pi^{\alpha_1\cdots\alpha_N 3} d_{\alpha_1\cdots\alpha_N\cdot}{}^{\lambda} - \pi^{\alpha_1\cdots\alpha_N\lambda} d_{\alpha_1\cdots\alpha_N\cdot}{}^{3})$$
$$+\sum_{N=2}^{\infty} (p^{\alpha_1\cdots\alpha_N 3}\lambda_{\alpha_1\cdots\alpha_N\cdot}{}^{\lambda} - p^{\alpha_1\cdots\alpha_N\lambda}\lambda_{\alpha_1\cdots\alpha_N\cdot}{}^{3}) = 0, \tag{11.30}$$

and

$$\chi^{(\alpha_1\cdots\alpha_n)\mu\lambda} - \chi^{(\alpha_1\cdots\alpha_n)\lambda\mu} + p^{\alpha_1\cdots\alpha_n\beta\lambda}\kappa_{\beta\cdot}{}^{\mu} - p^{\alpha_1\cdots\alpha_n\beta\mu}\kappa_{\beta\cdot}{}^{\lambda}$$
$$+\sum_{N=2}^{\infty} (\chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N\lambda} d_{\beta_1\cdots\beta_N\cdot}{}^{\mu} - \chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N\mu} d_{\beta_1\cdots\beta_N\cdot}{}^{\lambda})$$
$$+\sum_{N=2}^{\infty} (p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N\lambda}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{\mu} - p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N\mu}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{\lambda}) = 0, \tag{11.31}$$

$$\chi^{(\alpha_1\cdots\alpha_n)\lambda 3} - p^{\alpha_1\cdots\alpha_n\lambda} + p^{\alpha_1\cdots\alpha_n\beta 3}\kappa_{\beta\cdot}{}^{\lambda} - p^{\alpha_1\cdots\alpha_n\beta\lambda}\kappa_{\beta\cdot}{}^{3}$$
$$+\sum_{N=2}^{\infty} (\chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N 3} d_{\beta_1\cdots\beta_N\cdot}{}^{\lambda} - \chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N\lambda} d_{\beta_1\cdots\beta_N\cdot}{}^{3}) \tag{11.32}$$
$$+\sum_{N=2}^{\infty} (p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N 3}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{\lambda} - p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N\lambda}\lambda_{\beta_1\cdots\beta_N\cdot}{}^{3}) = 0.$$

We now use the Helmholtz free energy function A^*, where

$$A^* = U^* - T^*S^*, \tag{11.33}$$

and write

$$\rho A(a_{33})^{\frac{1}{2}} = \lambda A = \int\int kA^* \, d\theta^1 \, d\theta^2,$$
$$\rho S(a_{33})^{\frac{1}{2}} = \lambda S = \int\int kS^* \, d\theta^1 \, d\theta^2, \tag{11.34}$$

$$\lambda A^{\alpha_1 \cdots \alpha_n} = \int\int k\theta^{\alpha_1} \ldots \theta^{\alpha_n} A^* \, d\theta^1 \, d\theta^2,$$
$$\lambda S^{\alpha_1 \cdots \alpha_n} = \int\int k\theta^{\alpha_1} \ldots \theta^{\alpha_n} S^* \, d\theta^1 \, d\theta^2. \tag{11.35}$$

Then, from (10.6) and (11.33),

$$A = U - T_0 S - \sum_{N=1}^{\infty} S^{\alpha_1 \cdots \alpha_N} T_{\alpha_1 \cdots \alpha_N},$$
$$A^{\alpha_1 \cdots \alpha_n} = U^{\alpha_1 \cdots \alpha_n} - T_0 S^{\alpha_1 \cdots \alpha_n} - \sum_{N=1}^{\infty} S^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N} T_{\beta_1 \cdots \beta_N}, \tag{11.36}$$

so that (11.20) and (11.21) become

$$\begin{aligned} \lambda r - \lambda \left[\dot{A} + \dot{T}_0 S + T_0 \dot{S} + \sum_{N=1}^{\infty} S^{\alpha_1 \cdots \alpha_N} \dot{T}_{\alpha_1 \cdots \alpha_N} + \sum_{N=1}^{\infty} \dot{S}^{\alpha_1 \cdots \alpha_N} T_{\alpha_1 \cdots \alpha_N} \right] \\ + \bar{\pi}^{(\lambda\mu)} \eta_{\lambda\mu} + \bar{\pi}^{\lambda} \eta_{\lambda 3} + \bar{\pi} \eta_{33} + p^{\alpha i} \dot{\kappa}_{\alpha i} \\ + \sum_{N=2}^{\infty} \pi^{\alpha_1 \cdots \alpha_N i} \dot{d}_{\alpha_1 \cdots \alpha_N i} + \sum_{N=2}^{\infty} p^{\alpha_1 \cdots \alpha_N i} \dot{\lambda}_{\alpha_1 \cdots \alpha_N i} - \frac{\partial h}{\partial \theta} = 0, \end{aligned} \tag{11.37}$$

and

$$\begin{aligned} \lambda(r^{\alpha_1 \cdots \alpha_n} + R^{\alpha_1 \cdots \alpha_n}) - \lambda[\dot{A}^{\alpha_1 \cdots \alpha_n} + \dot{T}_0 S^{\alpha_1 \cdots \alpha_n} + T_0 \dot{S}^{\alpha_1 \cdots \alpha_n} \\ + \sum_{N=1}^{\infty} S^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N} \dot{T}_{\beta_1 \cdots \beta_N} + \sum_{N=1}^{\infty} \dot{S}^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N} T_{\beta_1 \cdots \beta_N}] \\ + \bar{\pi}^{(\lambda\mu)(\alpha_1 \cdots \alpha_n)} \eta_{\lambda\mu} + \bar{\pi}^{\lambda(\alpha_1 \cdots \alpha_n)} \eta_{\lambda 3} + \bar{\pi}^{(\alpha_1 \cdots \alpha_n)} \eta_{33} + p^{\alpha_1 \cdots \alpha_n \beta i} \dot{\kappa}_{\beta i} \\ + \sum_{N=2}^{\infty} \chi^{(\alpha_1 \cdots \alpha_n)\beta_1 \cdots \beta_N i} \dot{d}_{\beta_1 \cdots \beta_N i} + \sum_{N=2}^{\infty} p^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_N i} \dot{\lambda}_{\beta_1 \cdots \beta_N i} - \frac{\partial h^{\alpha_1 \cdots \alpha_n}}{\partial \theta} = 0. \end{aligned} \tag{11.38}$$

To complete the theory we should discuss entropy inequalities for rods on lines similar to those used in section 5 for shells. We leave aside this problem and, in the next section, study the thermoelastic theory of rods starting with the three dimensional results (6.1) and (6.2) together with definitions given in the present section.

12. THERMOELASTIC RODS

From (10.6), (10.11), (10.14), (11.1) and (11.34) we see that†

$$\begin{aligned} A &= A(T_{\beta_1\cdots\beta_M}, T_0, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1\cdots\alpha_N i}, \lambda_{\alpha_1\cdots\alpha_N i}), \\ A^{\alpha_1\cdots\alpha_n} &= A^{\alpha_1\cdots\alpha_n}(T_{\beta_1\cdots\beta_M}, T_0, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1\cdots\alpha_N i}, \lambda_{\alpha_1\cdots\alpha_N i}), \end{aligned} \tag{12.1}$$

for $M \geq 1$, $N \geq 2$, $n \geq 1$, where

$$\gamma_{ij} = a_{ij} - A_{ij}, \tag{12.2}$$

A_{ij} being the initial value of a_{ij}. With the help of (10.6), (10.11), (10.21), (10.22), (10.24)–(10.26), (11.16)–(11.18), (11.23)–(11.28), (11.34) and (11.35), by direct calculation, we obtain

$$\begin{aligned} S &= -\frac{\partial A}{\partial T_0}, \qquad S^{\alpha_1\cdots\alpha_n} = -\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial T_0} \\ S^{\beta_1\cdots\beta_N} &= -\frac{\partial A}{\partial T_{\beta_1\cdots\beta_N}}, \qquad S^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N} = -\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial T_{\beta_1\cdots\beta_N}}, \end{aligned} \tag{12.3}$$

for $n \geq 1$, $N \geq 1$, and

$$\bar{\pi} = 2\lambda\frac{\partial A}{\partial \gamma_{33}}, \qquad \bar{\pi}^{(\alpha_1\cdots\alpha_n)} = 2\lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial \gamma_{33}}, \tag{12.4}$$

$$\bar{\pi}^{\alpha} = 2\lambda\frac{\partial A}{\partial \gamma_{\alpha 3}}, \qquad \bar{\pi}^{\beta(\alpha_1\cdots\alpha_n)} = 2\lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial \gamma_{\beta 3}}, \tag{12.5}$$

$$\bar{\pi}^{(\lambda\mu)} = 2\lambda\frac{\partial A}{\partial \gamma_{\lambda\mu}}, \qquad \bar{\pi}^{(\lambda\mu)(\alpha_1\cdots\alpha_n)} = 2\lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial \gamma_{\lambda\mu}}, \tag{12.6}$$

$$\begin{aligned} p^{\alpha i} &= \lambda\frac{\partial A}{\partial \kappa_{\alpha i}}, \qquad p^{\alpha_1\cdots\alpha_n\beta i} = \lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial \kappa_{\beta i}}, \\ p^{\alpha_1\cdots\alpha_N i} &= \lambda\frac{\partial A}{\partial \lambda_{\alpha_1\cdots\alpha_N i}}, \qquad p^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N i} = \lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial \lambda_{\beta_1\cdots\beta_N i}}, \end{aligned} \tag{12.7}$$

for $n \geq 1$, $N \geq 2$, and

$$\pi^{\alpha_1\cdots\alpha_N i} = \lambda\frac{\partial A}{\partial d_{\alpha_1\cdots\alpha_N i}}, \qquad \chi^{(\alpha_1\cdots\alpha_n)\beta_1\cdots\beta_N i} = \lambda\frac{\partial A^{\alpha_1\cdots\alpha_n}}{\partial d_{\beta_1\cdots\beta_N i}}, \tag{12.8}$$

for $n \geq 1$, $N \geq 2$. In evaluating (12.5) and (12.6), we regard A and $A^{\alpha_1\cdots\alpha_n}$ as functions of γ_{33}, $\gamma_{\lambda 3}$ and $\frac{1}{2}(\gamma_{\lambda\mu}+\gamma_{\mu\lambda})$. Using these constitutive equations, the residual energy equations (11.37) and (11.38) reduce to

$$\lambda r - \lambda\left(T_0\dot{S} + \sum_{N=1}^{\infty} \dot{S}^{\alpha_1\cdots\alpha_N} T_{\alpha_1\cdots\alpha_N}\right) - \frac{\partial h}{\partial \theta} = 0, \tag{12.9}$$

$$\lambda(r^{\alpha_1\cdots\alpha_n} + R^{\alpha_1\cdots\alpha_n}) - \lambda\left(T_0\dot{S}^{\alpha_1\cdots\alpha_n} + \sum_{N=1}^{\infty} \dot{S}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_N} T_{\beta_1\cdots\beta_N}\right) - \frac{\partial h^{\alpha_1\cdots\alpha_n}}{\partial \theta} = 0. \tag{12.10}$$

To complete the theory we require constitutive equations for h, $h^{\alpha_1\cdots\alpha_n}$ and $R^{\alpha_1\cdots\alpha_n}$ and this will be discussed in Section 14.

† The free energy A and $A^{\alpha_1\cdots\alpha_n}$ also depend on A_{ij} and the initial values of other quantities specified in (12.1).

We observe that equations (11.23) and (11.29) determine $\pi^{\lambda\mu}$, equation (11.25) determines n^{λ}, (11.27) determines n^3 and (11.30) determines $\pi^{\lambda 3}$. Also we have the system of equations (11.24), (11.26), (11.28), (11.31) and (11.32), as well as the last relation in (11.19). Equations (11.24), (11.31), (11.32), $(12.6)_2$, (12.7) and $(12.8)_2$ determine $\chi^{(\alpha_1\cdots\alpha_n)\lambda i}$ and the last equation in (11.19) is then an identity. In view of (12.4), (12.5), (12.7) and (12.8) and the fact that A and $A^{\alpha_1\cdots\alpha_n}$ in (12.1) are evaluated from (11.35) with the help of (10.11), the equations (11.26) and (11.28) are satisfied identically in the general theory. From (11.19) and (10.24) we see that $\chi^{(\alpha_1\cdots\alpha_n)\lambda i}$ are components of the vector

$$\boldsymbol{\chi}^{(\alpha_1\cdots\alpha_n)\lambda} = \int\int \mathbf{T}_{\lambda}\theta^{\alpha_1}\ldots\theta^{\alpha_n}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 \qquad (n \geq 1). \tag{12.11}$$

The equations involving $\mathbf{n}, \mathbf{p}^{\alpha_1\cdots\alpha_n}, \boldsymbol{\pi}^{\alpha_1\cdots\alpha_n} (n \geq 1)$, together with the residual energy equations, provide a complete system of equations for determining the kinematic and temperature variables if boundary conditions are imposed over the end sections of the rod. The vectors (12.11) can then be found by a subsequent calculation. The relevant system of equations for the kinematic and temperature variables are summarized below. The equations of motion (11.5) and (11.8) in component form are

$$\frac{\delta n^i}{\delta\theta} + \lambda f^i = 0, \qquad \lambda q^{\alpha_1\cdots\alpha_n i} + \frac{\delta p^{\alpha_1\cdots\alpha_n i}}{\delta\theta} = \pi^{\alpha_1\cdots\alpha_n i}, \tag{12.12}$$

$$\begin{aligned}\pi^{\lambda\mu} - \pi^{\mu\lambda} + p^{\alpha\mu}\kappa_{\alpha\cdot}^{\ \lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}^{\ \mu} + \sum_{N=2}^{\infty}(\pi^{\alpha_1\cdots\alpha_N\mu}d_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\lambda} - \pi^{\alpha_1\cdots\alpha_N\lambda}d_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\mu})\\ + \sum_{N=2}^{\infty}(p^{\alpha_1\cdots\alpha_N\mu}\lambda_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\lambda} - p^{\alpha_1\cdots\alpha_N\lambda}\lambda_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\mu}) = 0,\end{aligned} \tag{12.13}$$

$$\begin{aligned}\pi^{\lambda 3} - n^{\lambda} + p^{\alpha 3}\kappa_{\alpha\cdot}^{\ \lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}^{\ 3} + \sum_{N=2}^{\infty}(\pi^{\alpha_1\cdots\alpha_N 3}d_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\lambda} - \pi^{\alpha_1\cdots\alpha_N\lambda}d_{\alpha_1\cdots\alpha_N\cdot}^{\qquad 3})\\ + \sum_{N=2}^{\infty}(p^{\alpha_1\cdots\alpha_N 3}\lambda_{\alpha_1\cdots\alpha_N\cdot}^{\qquad\lambda} - p^{\alpha_1\cdots\alpha_N\lambda}\lambda_{\alpha_1\cdots\alpha_N\cdot}^{\qquad 3}) = 0.\end{aligned} \tag{12.14}$$

The constitutive equations are

$$S = -\frac{\partial A}{\partial T_0}, \qquad S^{\beta_1\cdots\beta_N} = -\frac{\partial A}{\partial T_{\beta_1\cdots\beta_N}} \qquad (N \geq 1), \tag{12.15}$$

$$\bar{\pi} = 2\lambda\frac{\partial A}{\partial\gamma_{33}}, \qquad \bar{\pi}^{\alpha} = 2\lambda\frac{\partial A}{\partial\gamma_{\alpha 3}}, \qquad \bar{\pi}^{(\alpha\beta)} = 2\lambda\frac{\partial A}{\partial\gamma_{\alpha\beta}}, \tag{12.16}$$

$$p^{\alpha i} = \lambda\frac{\partial A}{\partial\kappa_{\alpha i}}, \tag{12.17}$$

$$p^{\alpha_1\cdots\alpha_N i} = \lambda\frac{\partial A}{\partial\lambda_{\alpha_1\cdots\alpha_N i}} \qquad (N \geq 2), \tag{12.18}$$

$$\pi^{\alpha_1\cdots\alpha_N i} = \lambda\frac{\partial A}{\partial d_{\alpha_1\cdots\alpha_N i}} \qquad (N \geq 2), \tag{12.19}$$

$$A = A(T_{\beta_1\cdots\beta_M}, T_0, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1\cdots\alpha_N i}, \lambda_{\alpha_1\cdots\alpha_N i}) \qquad (M \geq 1, N \geq 2), \tag{12.20}$$

where $\bar{\pi}^{(\alpha\beta)}$, $\bar{\pi}^{\alpha}$, $\bar{\pi}$ are defined in (11.23), (11.25) and (11.27). The function A must be written in a form which allows for the appropriate symmetries in $\gamma_{\alpha\beta}$, $d_{\alpha_1\cdots\alpha_N i}$ and $\lambda_{\alpha_1\cdots\alpha_N i}$. The residual energy equations are given by (12.9) and (12.10). We shall not usually be concerned with the equations which determine the vectors (12.11).

13. APPROXIMATION FOR RODS

Previously in I where temperature effects were only partly considered, a method of approximation was suggested in order to reduce the infinite set of equations for the kinematic quantities to finite form. As in the problem of shells, we have found that the approximation procedure is only partly satisfactory so we replace it by another here; but at this stage we make no approximation in the temperature.

We assume that the free energy function A in (12.20) can be represented by an approximate expression in terms of γ_{ij}, $\kappa_{\alpha i}$, T_0, $T_{\beta_1\cdots\beta_N} (N \geq 1)$ only. We do not solve the problem of how to determine this approximate form of A from the expression (12.20) which is obtained from the full three dimensional theory. Thus, we set

$$A = A(T_{\beta_1\cdots\beta_N}, T_0, \gamma_{ij}, \kappa_{\alpha i}), \tag{13.1}$$

approximately, where $A(T_{\beta_1\cdots\beta_N}, T_0, \gamma_{ij}, \kappa_{\alpha i})$ is a different function from that in (12.20). Using (12.18) and (12.19), it follows that†

$$p^{\alpha_1\cdots\alpha_N i} = 0, \qquad \pi^{\alpha_1\cdots\alpha_N i} = 0 \qquad (N \geq 2). \tag{13.2}$$

The equations $(12.12)_2$ are then satisfied if

$$q^{\alpha_1\cdots\alpha_N i} = 0 \qquad (N \geq 2). \tag{13.3}$$

The remaining equations $(12.12)_1$, (12.13) and (12.14) become

$$\frac{\delta n^i}{\delta\theta} + \lambda f^i = 0, \qquad \lambda q^{\alpha i} + \frac{\delta p^{\alpha i}}{\delta\theta} = \pi^{\alpha i}, \tag{13.4}$$

$$\pi^{\lambda\mu} - \pi^{\mu\lambda} + p^{\alpha\mu}\kappa_{\alpha\cdot}^{\ \lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}^{\ \mu} = 0, \qquad \pi^{\lambda 3} - n^{\lambda} + p^{\alpha 3}\kappa_{\alpha\cdot}^{\ \lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}^{\ 3} = 0. \tag{13.5}$$

Constitutive equations are

$$\begin{gathered}
n^3 - p^{\alpha 3}\kappa_{\alpha\cdot}^{\ 3} = 2\lambda\frac{\partial A}{\partial\gamma_{33}}, \\
n^{\alpha} - p^{\beta 3}\kappa_{\beta\cdot}^{\ \alpha} = \lambda\frac{\partial A}{\partial\gamma_{\alpha 3}}, \\
\pi^{\lambda\mu} + \pi^{\mu\lambda} - p^{\alpha\lambda}\kappa_{\alpha\cdot}^{\ \mu} - p^{\alpha\mu}\kappa_{\alpha\cdot}^{\ \lambda} = 4\lambda\frac{\partial A}{\partial\gamma_{\lambda\mu}}, \\
p^{\alpha i} = \lambda\frac{\partial A}{\partial\kappa_{\alpha i}}, \\
S = -\frac{\partial A}{\partial T_0}, \qquad S^{\beta_1\cdots\beta_N} = -\frac{\partial A}{\partial T_{\beta_1\cdots\beta_N}}.
\end{gathered} \tag{13.6}$$

† These results hold approximately, since they are obtained with the help of (13.1).

The residual energy equations still retain the forms (12.9) and (12.10) and constitutive equations are required for $R^{\alpha_1 \cdots \alpha_n}$, h and $h^{\alpha_1 \cdots \alpha_n}$. We omit all equations which determine the vectors (12.11).

Further remarks about the above approximate theory are made at the end of section 14 for the special case of straight rods.

14. LINEAR THEORY FOR STRAIGHT RODS

Starting with a theory of the form of section 13 for the case when $T_{\beta_1 \cdots \beta_N} = 0\,(N \geq 1)$, Green, Laws and Naghdi [10] have studied a linear theory of straight elastic rods.† Only small changes are required to allow for non zero $T_{\beta_1 \cdots \beta_N}$ so we summarize the main results and refer to [10] for further details. We denote the initial values of the vectors $\mathbf{a}_i$ by $\mathbf{A}_i$ and choose $\mathbf{A}_i$ to be an orthonormal set with $\mathbf{A}_3$ a unit vector along the rod. Thus

$$\mathbf{A}_i \cdot \mathbf{A}_j = \delta_{ij}. \tag{14.1}$$

If

$$\mathbf{r} = \theta \mathbf{A}_3 + \mathbf{u}, \qquad \mathbf{a}_i \equiv \mathbf{A}_i + \mathbf{b}_i$$

where $\mathbf{u}$, $\mathbf{b}_i$ are small, we can write

$$\begin{gathered} \mathbf{u} = u_i \mathbf{A}_i, \qquad \mathbf{b}_i = b_{ij}\mathbf{A}_j, \\ \mathbf{b}_3 = \frac{\partial \mathbf{u}}{\partial \theta}, \qquad b_{3i} = \frac{\partial u_i}{\partial \theta}, \\ \gamma_{ij} = b_{ij} + b_{ji}, \qquad \kappa_{ij} = \frac{\partial b_{ij}}{\partial \theta}, \end{gathered} \tag{14.2}$$

since there is no distinction now between upper and lower case indices. The initial rod is unstressed and at uniform temperature θ_0 and we suppose that T^* in (10.6) denotes temperature differences from θ_0. With the usual linearization, the equations of motion in Section 13 reduce, for a straight rod, to

$$\frac{\partial n_i}{\partial \theta} + \lambda f_i = \lambda \frac{\partial^2 u_i}{\partial t^2}, \tag{14.3}$$

$$\frac{\partial m_1}{\partial \theta} - n_2 + \lambda q_{23} = 0, \qquad \frac{\partial m_2}{\partial \theta} + n_1 - \lambda q_{13} = 0, \qquad \frac{\partial m_3}{\partial \theta} + \lambda(q_{12} - q_{21}) = 0, \tag{14.4}$$

$$\begin{gathered} \pi_{11} = \lambda q_{11} + \frac{\partial p_{11}}{\partial \theta}, \qquad \pi_{22} = \lambda q_{22} + \frac{\partial p_{22}}{\partial \theta}, \\ 2\pi_{12} = 2\pi_{21} = \lambda(q_{12} + q_{21}) + \frac{\partial}{\partial \theta}(p_{12} + p_{21}), \end{gathered} \tag{14.5}$$

where λ is now the initial density and

$$m_1 = p_{23}, \qquad m_2 = -p_{13}, \qquad m_3 = p_{12} - p_{21}. \tag{14.6}$$

† The work of Green, Laws and Naghdi was based on an exact theory of rods obtained by separate postulates and not deduced from three dimensional equations.

The constitutive equations are

$$n_3 = 2\lambda\frac{\partial A}{\partial \gamma_{33}}, \qquad n_\beta = \lambda\frac{\partial A}{\partial \gamma_{\beta 3}},$$

$$p_{\alpha i} = \lambda\frac{\partial A}{\partial \kappa_{\alpha i}}, \qquad \pi_{\alpha\beta}+\pi_{\beta\alpha} = 4\lambda\frac{\partial A}{\partial \gamma_{\alpha\beta}}, \tag{14.7}$$

$$S = -\frac{\partial A}{\partial T_0}, \qquad S^{\beta_1\cdots\beta_N} = -\frac{\partial A}{\partial T_{\beta_1\cdots\beta_N}},$$

where $S, S^{\beta_1\cdots\beta_N}$ denote entropy differences from initial values and A is a quadratic form in the variables

$$\gamma_{ij}, \kappa_{\alpha j}, T_0, T_{\beta_1\cdots\beta_N}. \tag{14.8}$$

Also for a rod which is symmetric for reflections along the directions $\mathbf{A}_1, \mathbf{A}_2$,

$$q_{\beta i} = l_{\beta i}-\alpha_\beta\frac{\partial^2 b_{\beta i}}{\partial t^2} \qquad (\beta \text{ not summed}), \tag{14.9}$$

where $l_{\beta i}$ are components of assigned director force and α_β are inertia coefficients. The residual energy equations reduce to

$$\lambda r-\lambda\theta_0\dot{S}-\frac{\partial h}{\partial \theta} = 0,$$

$$\lambda(r^{\alpha_1\cdots\alpha_n}+R^{\alpha_1\cdots\alpha_n})-\lambda\theta_0\dot{S}^{\alpha_1\cdots\alpha_n}-\frac{\partial h^{\alpha_1\cdots\alpha_n}}{\partial \theta} = 0. \tag{14.10}$$

Previously, Green, Laws and Naghdi [10] considered a quadratic form for the variables (14.8) with $T_{\beta_1\cdots\beta_N} = 0\ (N \geq 1)$, which was invariant under the transformations

$$\mathbf{a}_i \to \pm-\mathbf{a}_i, \qquad \mathbf{A}_i \to \pm-\mathbf{A}_i, \tag{14.11}$$

assuming T_0 is unaltered by such transformations. Here we deal with non zero values of $T_{\beta_1\cdots\beta_N}$ and assume that $T_{\beta_1\cdots\beta_N}$ are unaltered when $\theta \to -\theta$ and that

$$T_{\beta_1\cdots\beta_N} \to -T_{\beta_1\cdots\beta_N}, \tag{14.12}$$

when $\mathbf{a}_1 \to -\mathbf{a}_1, \mathbf{A}_1 \to -\mathbf{A}_1$ and an odd number of the indices take the value 1. Under the same transformation of vectors $\mathbf{a}_1, \mathbf{A}_1$

$$T_{\beta_1\cdots\beta_N} \to T_{\beta_1\cdots\beta_N}, \tag{14.13}$$

where an even number of indices take the value 1. Similarly, when $\mathbf{a}_2 \to -\mathbf{a}_2, \mathbf{A}_2 \to -\mathbf{A}_2$, we have

$$T_{\beta_1\cdots\beta_N} \to -T_{\beta_1\cdots\beta_N}, \tag{14.14}$$

where an odd number of indices take the value 2 and

$$T_{\beta_1\cdots\beta_N} \to T_{\beta_1\cdots\beta_N}, \tag{14.15}$$

where an even number of indices take the value 2.

In the rest of this section we restrict attention to the case in which

$$T_{\beta_1 \cdots \beta_N} = 0 \qquad (N \geq 2), \tag{14.16}$$

so that T_0, T_1, T_2 are the only surviving temperatures. With invariance under the transformations (14.11) and the subsequent conditions, the quadratic for A assumes the form

$$\begin{aligned} 2\lambda A = {} & k_1\gamma_{11}^2 + k_2\gamma_{22}^2 + k_3\gamma_{33}^2 + \tfrac{1}{4}k_4(\gamma_{12}+\gamma_{21})^2 + k_5\gamma_{23}^2 \\ & + k_6\gamma_{13}^2 + k_7\gamma_{11}\gamma_{22} + k_8\gamma_{11}\gamma_{33} + k_9\gamma_{22}\gamma_{33} \\ & + k_{10}\kappa_{11}^2 + k_{11}\kappa_{22}^2 + k_{12}\kappa_{12}^2 + k_{13}\kappa_{21}^2 + k_{14}\kappa_{12}\kappa_{21} \\ & + k_{15}\kappa_{23}^2 + k_{16}\kappa_{13}^2 + k_{17}\kappa_{11}\kappa_{22} + 2k_{18}T_0\gamma_{11} + 2k_{19}T_0\gamma_{22} \\ & + 2k_{20}T_0\gamma_{33} + k_{21}T_0^2 + 2k_{22}T_1\kappa_{13} + 2k_{23}T_2\kappa_{23} + k_{24}T_1^2 + k_{25}T_2^2. \end{aligned} \tag{14.17}$$

From (14.17) and the other equations of this section we can obtain expressions for stresses and entropy. The equations separate into four distinct groups, two concerned with flexure, one with torsion and one with logitudinal extension. The two sets of equations for flexure, including the equations of motion and energy equations†, are

$$\left.\begin{aligned} \frac{\partial n_1}{\partial \theta} + \lambda f_1 = \lambda\frac{\partial^2 u_1}{\partial t^2}, \qquad & \frac{\partial m_2}{\partial \theta} + n_1 - \lambda l_{13} + \lambda\alpha_1\frac{\partial^2 b_{13}}{\partial t^2} = 0, \\ n_1 = k_6(b_{13}+b_{31}), \qquad & m_2 = -k_{16}\frac{\partial b_{13}}{\partial \theta} - k_{22}T_1, \\ \lambda S^1 = -k_{22}\frac{\partial b_{13}}{\partial \theta} - k_{24}T_1, \qquad & b_{31} = \frac{\partial u_1}{\partial \theta}, \\ \lambda(r^1+R^1) - \lambda\theta_0\dot{S}^1 - & \frac{\partial h^1}{\partial \theta} = 0, \end{aligned}\right\} \tag{14.18}$$

$$\left.\begin{aligned} & \frac{\partial n_2}{\partial \theta} + \lambda f_2 = \lambda\frac{\partial^2 u_2}{\partial \theta^2}, \qquad \frac{\partial m_1}{\partial \theta} - n_2 + \lambda l_{23} - \lambda\alpha_2\frac{\partial^2 b_{23}}{\partial t^2} = 0, \\ & n_2 = k_5(b_{23}+b_{32}), \qquad m_1 = k_{15}\frac{\partial b_{23}}{\partial \theta} + k_{23}T_2, \\ & \lambda S^2 = -k_{23}\frac{\partial b_{23}}{\partial \theta} - k_{25}T_2, \qquad b_{32} = \frac{\partial u_2}{\partial \theta}, \qquad \lambda(r^2+R^2) - \lambda\theta_0\dot{S}^2 - \frac{\partial h^2}{\partial \theta} = 0. \end{aligned}\right\} \tag{14.19}$$

The equations for torsional motion of the rod are

$$\left.\begin{aligned} \frac{\partial m_3}{\partial \theta} + \lambda(l_{12}-l_{21}) &= \lambda\left(\alpha_1\frac{\partial^2 b_{12}}{\partial t^2} - \alpha_2\frac{\partial^2 b_{21}}{\partial t^2}\right), \\ 2\pi_{12} &= \frac{\partial}{\partial \theta}(p_{12}+p_{21}) + \lambda(l_{12}+l_{21}) - \lambda\left(\alpha_1\frac{\partial^2 b_{12}}{\partial t^2} + \alpha_2\frac{\partial^2 b_{21}}{\partial t^2}\right), \\ m_3 &= (k_{12}-\tfrac{1}{2}k_{14})\frac{\partial b_{12}}{\partial \theta} - (k_{13}-\tfrac{1}{2}k_{14})\frac{\partial b_{21}}{\partial \theta}, \\ \pi_{12} &= k_4(b_{12}+b_{21}), \\ p_{12}+p_{21} &= (k_{12}+\tfrac{1}{2}k_{14})\frac{\partial b_{12}}{\partial \theta} + (k_{13}+\tfrac{1}{2}k_{14})\frac{\partial b_{21}}{\partial \theta}, \end{aligned}\right\} \tag{14.20}$$

† The energy equations in (14.10) for $n \geqq 2$ are, in fact, only satisfied by an appropriate choice for $r^{\alpha_1 \cdots \alpha_n} (n \geqq 2)$.

and the equations for extensional motion of the rod are

$$\begin{aligned}
&\frac{\partial n_3}{\partial \theta}+\lambda f_3 = \lambda\frac{\partial^2 u_3}{\partial t^2},\\
&\pi_{11} = \frac{\partial p_{11}}{\partial \theta}+\lambda\left(l_{11}-\alpha_1\frac{\partial^2 b_{11}}{\partial t^2}\right),\\
&\pi_{22} = \frac{\partial p_{22}}{\partial \theta}+\lambda\left(l_{22}-\alpha_2\frac{\partial^2 b_{22}}{\partial t^2}\right),\\
&n_3 = 2k_8 b_{11}+2k_9 b_{22}+4k_3 b_{33}+2k_{20}T_0,\\
&\pi_{11} = 4k_1 b_{11}+2k_7 b_{22}+2k_8 b_{33}+2k_{18}T_0,\\
&\pi_{22} = 2k_7 b_{11}+4k_2 b_{22}+2k_9 b_{33}+2k_{19}T_0,\\
&p_{11} = k_{10}\frac{\partial b_{11}}{\partial \theta}+\frac{1}{2}k_{17}\frac{\partial b_{22}}{\partial \theta},\\
&p_{22} = k_{11}\frac{\partial b_{22}}{\partial \theta}+\frac{1}{2}k_{17}\frac{\partial b_{11}}{\partial \theta},\\
&\lambda S = -k_{18}\gamma_{11}-k_{19}\gamma_{22}-k_{20}\gamma_{33}-k_{21}T_0, \qquad \lambda r-\lambda\theta_0\dot{S}-\frac{\partial h}{\partial \theta} = 0.
\end{aligned} \tag{14.21}$$

Equations (14.20) and (14.21) have been given previously by Green, Laws and Naghdi and were derived from an exact system of equations which were not deduced from three dimensional equations by approximation. These authors also gave the system of equations (14.18) and (14.19) when $T_1 = T_2 = 0$. To complete the theory characterized by equations (14.18), (14.19) and (14.21) we require constitutive equations for R^1, R^2, h, h^1, h^2. These can be obtained by separate postulates but here we derive them from the three dimensional form of the heat conduction vector for a rod which is transversely isotropic with respect to its length. In view of the special system of vectors $\mathbf{A}_i$ chosen here for the straight rod in its initial state we can regard $\theta^1, \theta^2, \theta$ as a system of rectangular Cartesian coordinates so that

$$q^{*\alpha} = -\kappa\frac{\partial T^*}{\partial \theta^\alpha}, \qquad q^{*3} = -\kappa'\frac{\partial T^*}{\partial \theta} \tag{14.22}$$

where κ, κ' are constants. We assume that the temperature of the medium surrounding the rod is T_+, a function of θ, and we adopt the surface condition

$$h^* = H(T^*-T_+) \qquad [f(\theta^1,\theta^2) = 0], \tag{14.23}$$

where H is a constant. In view of (14.16), we recall that

$$T^* = T_0+\theta^1 T_1+\theta^2 T_2 \tag{14.24}$$

approximately, where T_0, T_1, T_2 are functions of θ and t.

From (10.15) and (14.23), we have

$$\lambda r = \iint kr^*\, d\theta^1\, d\theta^2 - H\oint (T_0-T_+-\theta^1 T_1-\theta^2 T_2)\, ds$$

where, since θ^1, θ^2 are rectangular Cartesian coordinates,

$$\mathrm{d}s = u^1\,\mathrm{d}\theta^2 - u^2\,\mathrm{d}\theta^1,$$

$\mathrm{d}s$ being a line element along the curve θ = constant, $f(\theta^1, \theta^2) = 0$. Recalling that the rod has symmetries about the directions θ^1, θ^2 it follows that

$$\lambda r = \lambda\bar{r} - Hl(T_0 - T_+), \tag{14.25}$$

where

$$\lambda\bar{r} = \iint kr^*\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \qquad l = \oint \mathrm{d}s, \tag{14.26}$$

and l is the length of the boundary curve of a normal section of the rod. Similarly, from (10.29), (10.31), (14.22) and (14.23) we see that

$$\lambda R^\alpha + \lambda r^\alpha = \lambda\bar{r}^\alpha - \kappa\Delta T_\alpha + H I'_\alpha T_\alpha \quad (\alpha \text{ not summed}), \tag{14.27}$$

where

$$\begin{gathered}\lambda\bar{r}^\alpha = \iint k\theta^\alpha r^*\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2,\\ \Delta = \iint \mathrm{d}\theta^1\,\mathrm{d}\theta^2, \qquad I'_\alpha = \oint (\theta^\alpha)^2\,\mathrm{d}s.\end{gathered} \tag{14.28}$$

Also, from (10.20), (10.27), (14.22) and (14.24) since here $h^* = q^{*3}$,

$$h = -\kappa'\Delta\frac{\partial T_0}{\partial\theta}, \qquad h^\alpha = -\kappa' I_{\alpha\alpha}\frac{\partial T_\alpha}{\partial\theta} \quad (\alpha \text{ not summed}), \tag{14.29}$$

where

$$I_{\alpha\alpha} = \iint \theta^\alpha\theta^\alpha\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 \quad (\alpha \text{ not summed}). \tag{14.30}$$

This completes the specification of all quantities appearing in the energy equations in (14.18), (14.19) and (14.21).

By comparing some exact solutions from the three dimensional linear elasticity with corresponding solutions predicted by the approximate theory, we can identify some of the elastic coefficients which occur in the approximate value of A in (14.17). In this manner, Green, Laws and Naghdi [10] have previously given values for the coefficients k_{15} and k_{16}. We defer to a future occasion the problem of finding values for the remaining coefficients in (14.17).

In view of the approximation made in the value of A, we expect that some of the identities in section 11 (mentioned in the paragraph after (2.10)) will not now be satisfied. In particular, the identity $(11.19)_4$ yields

$$\chi^{(\alpha_1)\alpha_2 i} + \chi^{(\alpha_2)\alpha_1 i} = 0. \tag{14.31}$$

Also, for the linear elastic rod, (11.32) gives

$$\chi^{(\alpha_1)\lambda 3} = p^{\alpha_1\lambda}, \tag{14.32}$$

approximately. Hence

$$p^{\alpha_1\alpha_2}+p^{\alpha_2\alpha_1} = 0, \tag{14.33}$$

or

$$p^{11} = p^{22} = 0, \qquad p^{12}+p^{21} = 0. \tag{14.34}$$

The first conditions in (14.34) provide the restrictions $k_{10} = k_{11} = k_{17} = 0$ on the equations (14.21) of the extensional theory while $(14.34)_2$ provide restrictions on the torsional motion governed by equations (14.20). The latter condition appears to lead to a theory for the torsion of a bar for which $k_{12} = k_{13}$, $\alpha_1 = \alpha_2$ and $b_{12}+b_{21} = 0$. We note here that if we include (14.34) as part of the approximation procedure, the resulting approximate theory is more restrictive than the corresponding results for a Cosserat curve, obtained by Green and Laws [4] from separate postulates. However, it does not appear essential to use (14.34).

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Абстракт—Рассматривается подробная разработка нелинейных термодинамических теорий стержней и оболочек, используя в качестве исходной точки, трехмерную теорию классической механики сплошной среды. Часть работы дополняет и расширяет предыдущую работу по этому вопросу Грина, Лявса и Нагхди (1). Изменяется, также, метод аппроксимации, используемый в (1), в виду того, что он оказывается достаточным только в некоторой степени для решения задачи. Обращается специальное внимание на неизотермические линейные теории упругих оболочек и прямых упругих стержней, полученные в предыдущих уравнениях.